Quantifying the effect of physical uncertainties in unsteady fluid-structure interaction problems*

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The long-term periodic limit cycle oscillation (LCO) response of unsteady fluid-structure interaction systems can be sensitive to physical input variations. Polynomial Chaos expansions can however fail to predict the effect of uncertainties in long-term time integration problems. In this paper, a non-intrusive Polynomial Chaos formulation for modeling the effect of uncertainties on the periodic response of dynamical systems is proposed. It is based on the application of Probabilistic Collocation (PC) onto a time-independent parametrization of the periodic response of the deterministic realizations, which is referred to as Probabilistic Collocation for limit cycle oscillations (PCLCO). The time-independent parametrization enables PCLCO to resolve the asymptotic stochastic behavior of dynamical systems successfully. Applications to a two-degree-of-freedom airfoil flutter model and the fluid-structure interaction of an elastically-mounted cylinder are presented.

I. Introduction

Confidence in computational predictions can be enhanced by including physical uncertainties in numerical simulations. The effect of uncertainties in the input data is nowadays relatively large due to reduction of numerical errors in the last decades. This is especially true for the prediction of the long-term response of dynamical systems, such as fluid-structure interaction problems, since dynamical systems often amplify input variability in time. The effect of uncertainty in fluid-structure interaction problems is of interest to engineers, since it can lead to an earlier onset of unstable flutter behavior which can lead to failure or fatigue damage of the structure. Flutter is the loss of dynamical stability at a critical dynamic pressure to a time periodic instability that can grow in an unbounded fashion. In practice, nonlinear systems exhibit a periodic response beyond the flutter point which is known as a limit cycle oscillation (LCO). It is known that the existence and the properties of LCO depend on input variations. Therefore, the complete analysis of LCO should include the quantification of the effects of physical input uncertainty.

Only parametric uncertainty given by a random variable is considered in this work, which can be modeled using the Polynomial Chaos expansion, which is a polynomial expansion in terms of independent random variables and deterministic coefficients. The deterministic coefficients can be solved for numerically by applying a Stochastic Galerkin method or a Probabilistic Collocation (PC) method. The Stochastic Galerkin approach employs a Galerkin projection in probability space, which results in a coupled system of deterministic equations. Probabilistic Collocation approaches are non-intrusive methods which collocate the uncertainty quantification problem in deterministic problems for various parameter values. Usually, Gauss quadrature collocation is employed.

Due to the superior exponential convergence properties of the Polynomial Chaos expansion, large efficiency gains have been demonstrated compared to Monte Carlo simulations. However, it has been found that the Polynomial Chaos expansion can have difficulty predicting the effect of input uncertainty in long-term time integration problems accurately, especially when the response frequencies are affected. In general

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a high Polynomial Chaos degree is required, since the degree increases in time with increasing nonlinearity of the response surface. A multi-element generalized Polynomial Chaos can extend the valid integration time.\textsuperscript{17} Approximations with Wiener-Haar expansions\textsuperscript{9,14} and Fourier Chaos expansions\textsuperscript{12} seem to be more accurate alternatives. However, Polynomial Chaos approximations fail asymptotically due to the growing nonlinearity of the response surface in time.

In this paper a Probabilistic Collocation formulation for quantifying the effect of input uncertainty on long-term limit cycle oscillation response (PCLCO) is proposed. In PCLCO, Probabilistic Collocation is applied to a time-independent parametrization of the deterministic realizations of the periodic response for various parameter values. The limit cycle oscillations are parametrized by the frequency, the relative phase, the amplitude, a reference value and the normalized period. Due to the time-independent parametrization the accuracy of the PCLCO approximation is independent of time, which enables it to resolve the asymptotic stochastic behavior of dynamical systems. In practice the error can slightly increase with time due to numerical integration errors. Furthermore, the parameters describing the periodic response are in practice often smooth functions of the uncertain input, which require a relatively low Polynomial Chaos order description. A Stochastic Collocation approach is used, since it can more effectively and cheaply approximate the functionals describing the limit cycle oscillation than a Stochastic Galerkin method.

PCLCO is in this work applied to problems with only a single random variable and one main frequency. The formulation needs further extension to problems with higher dimensionality of probability space using a multi-dimensional Stochastic Collocation approach and problems with more than one main frequency. Using this approach one may need to ensure that periodic solutions exist for the relevant input parameter range, which may not be trivial. One may also need to ensure that the parameters describing the periodic response are indeed smooth functions of the uncertain input. More complex real world problems can contain singularities in the response in probability space, for which an extension to a Multi-Element Stochastic Collocation formulation might be used.

The method is applied to a two-degree-of-freedom airfoil flutter model and the fluid-structure interaction of an elastically mounted cylinder. For the flutter model it is demonstrated that the Polynomial Chaos expansion fails to predict the correct stochastic behavior after short-term time integration by comparison with Monte Carlo simulations. Results for PCLCO show that the PCLCO formulation is able to resolve the long-term stochastic behavior of limit cycle oscillations and the stochastic transient successfully. Short-term results for which the deterministic samples are in their transient are however predicted less accurately, due to the periodic reconstruction used in PCLCO. Since Probabilistic Collocation can effectively model the short-term behavior, a combination of Probabilistic Collocation and PCLCO is employed to propagate the uncertainty through a fluid-structure interaction simulation of an elastically-mounted cylinder with an uncertain free stream velocity. For short-term integration in the transient part of the deterministic realizations Probabilistic Collocation is applied and PCLCO is used to resolve the stochastic transient behavior and the long-term stochastic response.

The Probabilistic Collocation approach for limit cycle oscillations is introduced in section II applied to the two-degree-of-freedom flutter model. The elastically-mounted cylinder is considered in section III. The paper is concluded in section IV.

II. Probabilistic Collocation for limit cycle oscillations

In this section PCLCO is introduced for a commonly used two-degree-of-freedom model of airfoil flutter. Such flutter models are standard engineering tools in flutter analysis instead of full unsteady fluid-structure interaction simulations. It is established in section II.B that Probabilistic Collocation has difficulty resolving the long-term stochastic solution for this application.

II.A. Airfoil flutter

A two-degree-of-freedom model for the pitch and plunge motion of a rigid airfoil is used to simulate airfoil flutter, see Figure 2(a). The model has for example been studied deterministically by Lee et al.\textsuperscript{7,8} and stochastically using Fourier Chaos by Millman et al.\textsuperscript{12} See appendix A for a detailed description of the model.

The ratio of uncoupled plunging and pitching modes natural frequencies $\bar{\omega}$ is assumed to be uncertain described by a lognormal distribution. The mean value is $\mu_{\bar{\omega}} = 0.2$ with a coefficient of variation of $CV_{\bar{\omega}} =$
10%. For the relevant range of \( \bar{\omega} \) for this settings the system exhibits a periodic response. The effect of the input uncertainty on the mean and variance of the pitch angle \( \alpha \) is considered. Results for the plunge deflection \( \zeta \) are qualitatively similar.

Three time series realizations of \( \alpha \) for a varying value of \( \bar{\omega} \), which are used in the Probabilistic Collocation and PCLCO approaches, are shown in Figure 3. The deterministic time series show a periodic limit cycle oscillation response after a transient for \( \tau < 100 \). The uncertainty in \( \bar{\omega} \) clearly affects the amplitude and the frequency of the response \( \alpha \). The effect on the frequency results in an increasing phase difference between the realizations. Uncertainty can in general also affect the shape of the periodic motion and the equilibrium position.

The mean and the variance of \( \alpha \) are given in Figures 4(a) and 4(b) as function of time for Monte Carlo simulation compared to Probabilistic Collocation and PCLCO. In contrast to the periodic realizations, Monte Carlo results based on 1000 uniform samples show a decaying oscillation to zero for the mean pitch angle after transient behavior for \( \tau < 100 \), see Figure 4(a). The transient of the mean corresponds to the transient part of the deterministic response. The decaying oscillation is caused by the effect of the uncertainty on the frequency of the realizations. Due to the increasing phase difference in time, realizations of pitch angles with opposite signs increasingly cancel each other.

For the variance of \( \alpha \) Monte Carlo simulation results in a stochastic transient till \( \tau = 1000 \), after which it approaches the steady value of \( 1.96 \cdot 10^{-2} \), see Figure 4(b). The transient domain of the variance is not limited to that of the deterministic realizations. In the transient part the variance shows an oscillation with an initially increasing amplitude until it approaches the steady value. Due to the bounded amplitude of the periodic realizations, the variance is limited to a finite asymptotic value.

II.B. Probabilistic Collocation

Probabilistic Collocation is reviewed below and it is demonstrated that the method has difficulty resolving the long-term stochastic solution of the airfoil flutter problem.

II.B.1. Probabilistic Collocation framework

Probabilistic Collocation approaches\(^2\)\(^,\)\(^10\)\(^,\)\(^11\) are based on collocating the stochastic problem in Gauss quadrature points in the probability space. Suitable Gauss points for approximating the stochastic moments of the response are the zeros of polynomials orthogonal with respect to the probability density of the uncertain input parameter \( a(\omega) \) with \( \omega \in \Omega \). The set of outcomes of the probability space \((\Omega, \mathcal{F}, P)\) is denoted by \( \Omega, \mathcal{F} \subset 2^\Omega \) is the \( \sigma \)-algebra of events and \( P \) is a probability measure. An uncertain variable \( u(x, t, \omega) \), with \( x \in \mathbb{R}^d \) and \( t \in T \), is then approximated in the Probabilistic Collocation method as

\[
u(x, t, \omega) = \sum_{k=1}^{N} u_k(x, t) l_k(a(\omega)), \tag{1}\]

where \( N \) is the number of collocation points \( \{a_k\}_{k=1}^{N} \) in probability space and \( N - 1 \) is the Polynomial Chaos order of the approximation (1). The collocation points \( \{a_k\}_{k=1}^{N} \) are the zeros of the polynomial \( \pi_{N+1}(a) \), where \( \{\pi_i(a)\}_{i=0}^{N+1} \) is the set of polynomials up to order \( N + 1 \) orthogonal with respect to the probability density function \( p_\alpha(a) \) of the uncertain input parameter \( a(\omega) \). For several “standard” input distributions the polynomials \( \{\pi_i(a)\}_{i=0}^{N+1} \) are (scaled) classical polynomials\(^{16} \) of which the roots are tabulated to full accuracy. For other input distributions the collocation points can be computed numerically.\(^10\)

The deterministic coefficients \( \{u_k(x, t)\}_{k=1}^{N} \) in (1) are then the deterministic realizations for the parameter values \( \{a_k\}_{k=1}^{N} \). The basis polynomials \( \{l_k(a)\}_{k=1}^{N} \) of the expansion (1) are Lagrange polynomials with respect to the collocation points \( \{a_k\}_{k=1}^{N} \) for which holds

\[
l_k(a_j) = \delta_{jk}, \quad j, k = 1, \ldots, N, \tag{2}\]

where \( \delta_{jk} \) is the Kronecker delta. Multidimensional collocation points can be obtained from tensor products of the one-dimensional collocation points or a sparse grid approach.\(^{21} \)

II.B.2. Probabilistic Collocation applied to airfoil flutter

In Figures 4(a) and 4(b) the results of Probabilistic Collocation are given, based on the three realizations \((N = 3)\) for the Gauss collocation points in probability space shown in Figure 3. Probabilistic Collocation
gives an accurate approximation of the mean for $\tau < 800$, also in the transient part $\tau < 100$, see Figure 4(a). For higher values of the non-dimensional time $\tau$, Probabilistic Collocation does not predict the asymptotic damped oscillation. This results in an inaccurate prediction of the effect of the uncertainty on the long-term response.

For the approximation of the variance in Figure 4(b), Probabilistic Collocation also fails to give an accurate prediction for long-term integration $\tau > 500$. For short-term time integration and the deterministic transient part $\tau < 100$ the Probabilistic Collocation predictions match the Monte Carlo results.

In long-term time integration of unsteady problems subject to uncertainty, the response surface $u(x, t, \omega)$ can be an increasingly nonlinear function of the uncertain input parameter $a(\omega)$. In that case, the global Polynomial Chaos representation in a Probabilistic Collocation approach is not adequate for approximating the response surface. Increasing the Polynomial Chaos order in Probabilistic Collocation or the number of elements in the multi-element generalized Polynomial Chaos extends the valid integration time, but it does not improve the accuracy significantly. The Polynomial Chaos approximation fails asymptotically due to the growing nonlinearity of the response surface in time.

II.C. Time-independent parametrization in PCLCO

Below the formulation of Probabilistic Collocation for limit cycle oscillations is given and applied to the airfoil flutter problem.

II.C.1. PCLCO formulation

To predict the effect of input uncertainty on the long-term periodic response, in this paper, Probabilistic Collocation is applied to a time-independent parametrization of the deterministic realizations instead of to the time-dependent realizations themselves. As mentioned before, input uncertainty affects the frequency $f(x, \omega)$, the relative phase $\phi(x, \omega)$, the amplitude $A(x, \omega)$, the reference value $u_0(x, \omega)$, and the normalized period $u_{\text{period}}(x, \tau, \omega)$, with $\tau \in [0, 2\pi]$, of the response. The periodic behavior of the realizations can be parametrized by these time-independent quantities instead of by time itself.

The time series realizations $u_k(x, t)$ of (1) for the $N$ collocation points $\{a_k\}_{k=1}^N$ in probability space result in $N$ realizations of the frequency $f_k(x)$, the phase $\phi_k(x)$, the amplitude $A_k(x)$, the reference value $u_0_k(x)$ and the normalized period $u_{\text{period}}(x, \tau)$, where see Figure 1:

- the frequency $f_k(x)$ is defined as the inverse of the period length, which is the smallest time $t_{\text{period}}(x) > 0$ for which holds in the asymptotic region $u_k(x, t + t_{\text{period}}(x)) = u_k(x, t)$;
- the relative phase $\phi_k(x)$ of the time series realizations is defined as the phase of the oscillation at $t = t_{\text{final}}$ with respect to the time of the latest maximum $t_{\text{max}}(x)$ by $\phi_k(x) = n_{\text{period}}(x) + (t_{\text{max}} - t_{\text{max}}(x))f_k(x)$ with $n_{\text{period}}(x)$ the integer number of completed cycles;
- the amplitude $A_k(x)$ is equal to half the difference between the minimum and the maximum of the period $A_k(x) = \frac{1}{2}(u_{\text{max}}(x) - u_{\text{min}}(x))$;
- the reference value $u_0_k(x)$ is chosen to be the average of the minimum and maximum of the period $u_0_k = \frac{1}{2}(u_{\text{min}}(x) + u_{\text{max}}(x))$.

These values are obtained from the last full period of a chosen sufficient long integration time. The Polynomial Chaos approximation of the parametrization $f(x, \omega)$, $\phi(x, \omega)$, $A(x, \omega)$, and $u_0(x, \omega)$ is then, for example for $f(x, \omega)$, described by the Probabilistic Collocation formulation (1)

$$f(x, \omega) = \sum_{k=1}^{N} f_k(x)l_k(a(\omega)).$$

(3)

The PCLCO approximation of the response is given by substitution of (3) into a parametrized description of the response given by

$$u(x, t, \omega) = u_0(x, \omega) + A(x, \omega)u_{\text{period}}(x, \tau(x, \omega), \omega),$$

(4)

with $\tau(x, \omega) = 2\pi(\phi(x, \omega) + (t - t_{\text{final}})f(x, \omega)) \mod 2\pi$. For this expression also an expansion of the normalized period $u_{\text{period}}(x, \tau, \omega)$ similar to (3) is required. The period of the deterministic realizations
discretize the normalized period. The normalized period $u_{\text{period}}(x, t_k(x))$ with $t_k(x) = \lbrack 0, t_{\text{period}}(x) \rbrack$ is extracted from the realizations $u_k(x, t)$. The normalized period $u_{\text{period}}(x, \tau)$, with $\tau \in [0, 2\pi]$, is obtained by scaling the periods $u'_{\text{period}}(x, t'_k(x))$ by their frequency $f_k(x)$, amplitude $A_k(x)$, and reference value $u_{0k}(x)$

$$u_{\text{period}}(x, \tau) = \frac{1}{x_k(x)} \left( u'_{\text{period}} \left( x, \frac{\tau}{2\pi f_k(x)} \right) - u_{0k}(x) \right), \tag{5}$$

with $k = 1, \ldots, N$ and $\tau \in [0, 2\pi]$. The Polynomial Chaos approximation of $u_{\text{period}}(x, \tau, \omega)$ based on the realizations $u_{\text{period}}(x, \tau)$ is given by

$$u_{\text{period}}(x, \tau, \omega) = \sum_{k=1}^{N} u_{\text{period}}(x, \tau) A_k(\omega). \tag{6}$$

In practice $u_{\text{period}}(x, \tau, \omega)$ is determined at $n_\tau$ discrete angles $\{\tau_j\}_{j=1}^{n_\tau} \in [0, 2\pi]$ and interpolation can be employed to obtain $u_{\text{period}}(x, \tau, \omega)$ from $\{u_{\text{period}}(x, \tau_j, \omega)\}_{j=1}^{n_\tau}$. One could also use a Fourier transform to discretize the normalized period.

The uncertainty distribution of the solution $u(x, t, \omega)$ at a certain time $t$ and the time series of the mean and variance are reconstructed using (4). It results in a non-polynomial approximation of the response surface $u(x, t, \omega)$. This formulation of Probabilistic Collocation for limit cycle oscillations is capable of resolving the effect of the uncertain input parameter $\omega$ on the long-term stochastic response. If the parametrization with $f(x, \omega)$, $\phi(x, \omega)$, $A(x, \omega)$, $u_0(x, \omega)$, and $u_{\text{period}}(x, \tau, \omega)$ depends not too nonlinearly on the uncertain input parameter $\omega$, the Polynomial Chaos order of the approximation of the long-term behavior can be relatively small. In pseudo algorithmic form PCLCO can be represented as

1. Solve $N$ deterministic problems for the parameter values corresponding to the $N$ collocation points in probability space;
2. Extract $f_k(x)$, $\phi_k(x)$, $A_k(x)$, $u_{0k}(x)$, and $u_{\text{period}}(x, \tau)$ for $k = 1$ to $k = N$ from the $N$ deterministic solutions;
3. Construct the global polynomial approximations $f(x, \omega)$, $\phi(x, \omega)$, $A(x, \omega)$, $u_0(x, \omega)$, and $u_{\text{period}}(x, \tau, \omega)$ using (3) and (6);
4. Substitute $f(x, \omega)$, $\phi(x, \omega)$, $A(x, \omega)$, $u_0(x, \omega)$, and $u_{\text{period}}(x, \tau, \omega)$ in (4) to find the approximation of the response $u(x, t, \omega)$.

The distribution function is then given by sorting the function $u-\omega$, with $\omega \in [0, 1]$, to a monotonically increasing reconstruction.

II.C.2. PCLCO applied to airfoil flutter

It can be seen in Figures 4(a) and 4(b) that PCLCO based on only three deterministic solves ($N = 3$) succeeds in predicting the stochastic long-term time integration results and the asymptotic values for the mean and the variance accurately. Only for $\tau < 100$ the solution is not accurately resolved, since PCLCO does not model the transient part of the deterministic samples. In Figure 3 it can be seen that the reconstruction of the periodic response of the deterministic samples by PCLCO continues for all $\tau$, while the time-dependent realizations exhibit a transient behavior for $\tau < 100$.

PCLCO and Probabilistic Collocation therefore seem complementary, where a Probabilistic Collocation post-processing should be used for the initial time interval in which the deterministic samples exhibit transient behavior. To the long-term periodic behavior of the deterministic samples after their transient, PCLCO post-processing should be applied. This combined approach is demonstrated in the next test problem.

III. Elastically mounted cylinder

The two-dimensional fluid-structure interaction problem of an elastically-mounted circular cylinder in a laminar Navier-Stokes flow with an uniform free stream is considered in this section, see Figure 2(b). The cylinder is only free to move in the cross flow $y$-direction, in which the structural stiffness is modeled by a
linear spring. The angular natural frequency of the structure in the y-direction is chosen to be \( \omega_n = \sqrt{0.1} \approx 0.316 \). The inflow velocity \( V \) is assumed to be uncertain described by a lognormal distribution. The mean value of the velocity \( \mu_V = 0.3 \) corresponds to a Reynolds number of \( Re = 1000 \). For the relevant range of Reynolds numbers for a coefficient of variation \( CV_V = 10\% \), the frequency of the periodic fluid motion is typically given by a Strouhal number of \( St = fd/V = 0.2 \), which corresponds to an angular frequency of \( \omega_{\text{flow}} = 0.38 \).

The circular spatial domain has a diameter of \( 40d \). An Arbitrary Lagrangian-Eulerian formulation is employed to couple the fluid mesh with the movement of the structure. For the spatial discretization a second-order finite volume method is employed. Time integration is performed using a BDF-2 method with a stepsize of \( \Delta t = 0.25 \) until \( t = 250 \). Initially the flow field is uniform and the cylinder has an initial deflection of \( y_{\text{init}} = 0.5d \).

A combination of PCLCO and Probabilistic Collocation is used to solve for the stochastic response of the cylinder in the whole time domain. For the short-term integration in the transient part of the deterministic realizations Probabilistic Collocation is applied. PCLCO is employed for resolving the stochastic transient behavior and the long-term stochastic response. In Figures 5(a) and 5(b) the evolution of the mean and the variance of the cylinder displacement \( y(t, \omega) \) is shown. To demonstrate the convergence of the combined approach for short-term and long-term integration, the approximations for \( N = 2 \) to \( N = 4 \) are shown. Probabilistic Collocation is applied to the deterministic samples in an initial time interval starting at \( t = 0 \). Starting from the time where the PCLCO and Probabilistic Collocation approximations match, the PCLCO approach is applied. These points are in Figures 5(a) and 5(b) denoted by the symbols.

A similar behavior of the mean and the variance can be seen as for the previous test problem. The mean is a decaying oscillation after the transient part of the deterministic realizations. The variance approaches an asymptotic value of approximately \( 9.6 \cdot 10^{-2} \) after an oscillatory stochastic transient, which extends beyond the deterministic transient.

In the initial time interval Probabilistic Collocation gives a converged solution already for the low order approximations, which is demonstrated by the coinciding approximations for \( N = \{2,3,4\} \). PCLCO also shows a converging solution, especially for the long term integration results \( t > 150 \). In the stochastic transient \( t \in [50,150] \) the results of PCLCO seem to converge less rapidly.

IV. Conclusions

A Probabilistic Collocation (PC) formulation for modeling the long-term stochastic behavior of limit cycle oscillations (PCLCO) is proposed. In PCLCO, Probabilistic Collocation is applied to a time-independent parametrization of the periodic time series realizations at collocation points in probability space. Due to its independence of time the PCLCO formulation is capable of modeling the long-term stochastic behavior of dynamic systems. The periodic limit cycle oscillation (LCO) realizations are parametrized using the frequency, relative phase, amplitude, reference value and normalized period. Numerical results are presented for an airfoil flutter model and the flow around an elastically-mounted cylinder. For the flutter model it is demonstrated that the Polynomial Chaos expansion fails to predict the correct stochastic behavior after short-term time integration. Results for PCLCO show that the PCLCO formulation is able of resolving the long-term stochastic behavior of limit cycle oscillations and the stochastic transient successfully. Short-term results for which the deterministic samples are in their transient is however not predicted accurately, due to the periodic reconstruction used in the PCLCO technique. Since Probabilistic Collocation can model the short-term behavior, a combination of Probabilistic Collocation and PCLCO is demonstrated for the elastically-mounted cylinder.

A. Two-degree-of freedom airfoil flutter model

The aerelastic equations of motion of the rigid airfoil with cubic restoring springs in both pitch and plunge are given as\(^8\)

\[
\xi'' + \frac{x_a C}{\pi \mu} \xi'' + 2 \xi \frac{\bar{w}}{U_c^2} \xi' + \left( \frac{\bar{w}}{U_c} \right)^2 \left( \xi + \beta \xi \xi^3 \right) = -\frac{1}{\pi \mu} C_L(\tau), \tag{7}
\]

\[
\frac{x_a C}{\pi \mu} \xi'' + \frac{C}{\pi \mu} \xi'' + 2 \xi \frac{\bar{w}}{U_c^2} \xi' + \frac{1}{U_c^2} \left( \alpha + \beta_\alpha \alpha^3 \right) = -\frac{2}{\pi \mu \alpha^2} C_M(\tau), \tag{8}
\]
where $\alpha$ is the pitch angle, $\xi = h/b$ is the non-dimensional plunge displacement of the elastic axis, with $b = c/2$ the half-chord, $\beta_1$ and $\beta_2$ are the nonlinear spring constants, $r_\alpha$ is the radius of gyration about the elastic axis, and $\zeta_1$ and $\zeta_2$ are the viscous damping coefficients in plunge and pitch, respectively. The ratio of natural frequencies is $\bar{\omega} = \omega_1/\omega_2$, where $\omega_1$ and $\omega_2$ are the natural frequencies of the uncoupled plunging and pitching modes, respectively. Non-dimensionalized time and windspeed are respectively defined as $\tau = Ut/b$ and $U^* = U/(b\bar{\omega}_1)$. The expressions for the aerodynamic force and moment coefficients, $C_L(\tau)$ and $C_M(\tau)$ are given by Fung as

$$
C_L(\tau) = \pi(\xi'' - a_h\alpha'' + \alpha') + 2\pi \left\{ \alpha(0) + \xi'(0) + \left[ \frac{1}{2} - a_h \right] \alpha'(0) \right\} \phi(\tau) + 
+ 2\pi \int_0^\tau \phi(\tau - \sigma) \left[ \alpha'(\sigma) + \xi''(\sigma) + \left( \frac{1}{2} - a_h \right) \alpha''(\sigma) \right] d\sigma, 
$$

$$
C_M(\tau) = \pi \left( \frac{1}{2} + a_h \right) \left\{ \alpha(0) + \xi'(0) + \left( \frac{1}{2} - a_h \right) \alpha'(0) \right\} \phi(\tau) + 
+ \pi \left( \frac{1}{2} + a_h \right) \int_0^\tau \phi(\tau - \sigma) \left\{ \alpha'(\sigma) + \xi''(\sigma) + \left( \frac{1}{2} - a_h \right) \alpha''(\sigma) \right\} d\sigma + 
+ \frac{\pi}{2} a_h (\xi'' - a_h\alpha'') - \left( \frac{1}{2} - a_h \right) \frac{\pi}{2} \alpha' - \frac{\pi}{16} \alpha'', 
$$

where the elastic axis is located at a distance $a_h b$ from mid-chord, the mass center is located at a distance $x_\alpha b$ from the elastic axis and $\phi(\tau)$ is the Wagner function

$$
\phi(\tau) = 1 - \psi_1 e^{-\varepsilon_1 \tau} - \psi_2 e^{-\varepsilon_2 \tau},
$$

with the constants $\psi_1 = 0.165$, $\psi_2 = 0.335$, $\varepsilon_1 = 0.0455$ and $\varepsilon_2 = 0.3$ given by Jones. Based on (7) to (11), a set of first-order ordinary differential equations for the motion of the airfoil can be derived. These equations are integrated numerically until $\tau = 2000$ using the explicit fourth-order Runge-Kutta method with a time step of $\Delta \tau = 0.1$, which is approximately $1/256$ of the smallest period. The following parameter values are used: $U^* = 6.6$, $\mu = 100$, $a_h = -0.5$, $x_\alpha = 0.25$, $r_\alpha = 0.5$ and $\zeta_\alpha = \zeta_1 = 0.8$. The nonlinear torsional spring stiffness parameter is set to $\beta_\alpha = 3$, for which the system exhibits a limit cycle oscillation.

References


American Institute of Aeronautics and Astronautics
Figure 1. Definition of parameters to describe limit cycle oscillation in PCLCO.

Figure 2. Unsteady fluid-structure interaction test problems.
Figure 3. The three deterministic realizations and their PCLCO reconstruction at the collocation points in probability space for the airfoil flutter model.

Figure 4. Mean and variance of the pitch angle by Monte Carlo (MC), PCLCO and Probabilistic Collocation (PC) for the two-degree-of-freedom airfoil flutter model.
Figure 5. Mean and variance of the deflection of the elastically-mounted cylinder for the combination of PCLCO and Probabilistic Collocation (PC).