Uncertainty quantification for laminar-turbulent transition prediction in RANS turbomachinery applications

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The effect of physical variability and model uncertainty on laminar-turbulent transition in transonic gas turbine compressors is computed using Stochastic Collocation methods. Aleatoric and epistemic uncertainties are considered in an adiabatic flat plate validation and turbine guide vane simulations. The computational results show that the variability has significant impact on the transition location for the turbine guide vane simulations and, consequently, on the reliability of the model. The model uncertainty accounts to a large extent for the difference between the deterministic simulation and the experiments. The results from the Simplex Stochastic Collocation method show a more robust convergence than those of Stochastic Collocation based on Clenshaw–Curtis quadrature.

I. Introduction

In turbomachines and especially in aircraft engines the Reynolds numbers that determine the evolution of the boundary layers are relatively low, hence a large part of the flow along the blade surfaces is laminar or transitional. Bypass transition is the dominant form of transition in turbomachinery due to the high turbulence levels, e.g., generated by upstream blade rows. The boundary layer development, losses, efficiency, and heat transfer are greatly affected by the laminar-to-turbulent transition. Therefore, the ability to accurately predict the transition process is crucial for the design of efficient and reliable machines.

Considerable effort has been spent in the past two decades to develop transition models for engineering applications to predict transitional boundary layers for various kinds of flows. In general, these models rely entirely on empirical correlations obtained from existing data sets for simple flow configurations. Unfortunately, for complex flows there is only limited experimental data available that can be incorporated to calibrate these models. Furthermore, in complex flow configurations the lack of knowledge of the exact inlet boundary conditions adds a second type of uncertainty. Previous uncertainty analysis studies have shown that transonic gas turbine compressors are sensitive to these uncertainties.

Therefore, this work focuses on the quantification of uncertainties due to the lack of knowledge of the physical processes associated with boundary layer transition (also called epistemic uncertainty or model form uncertainty) and the uncertainties of turbulence inlet conditions for complex flow situations (also called aleatory or irreducible uncertainties associated with boundary conditions in general). The epistemic uncertainties as well as the aleatory uncertainties are identified and addressed in the simulations of a transonic turbine guide vane.

The flow is modeled using the Reynolds-Averaged Navier-Stokes (RANS) equations closed with the \( k - \omega \) shear stress transport (SST) turbulence model\(^7\) and the transition is modeled using the \( \gamma - \tilde{R}_{eq}t \) correlation based transition model of Langtry and Menter.\(^5\) The correct implementation of these models is first validated for commonly used transition benchmark cases of a flow over an adiabatic flat plate, with and without imposed pressure gradient.\(^13\) The models are then applied to the boundary layer transition on the

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transonic turbine guide vane of the VKI test cases MUR241 and MUR235. The Uncertainty Quantification (UQ) method used is Stochastic Collocation\textsuperscript{2,18} (SC) with Clenshaw-Curtis (CC) quadrature points. It propagates the input uncertainties in the inlet turbulence intensity $Tu_{in}$, turbulence Reynolds number $Re_T$, and the uncertainties associated with the transition model through the computational domain. The results for the case with three uncertainties in $Re_T$ and two epistemic parameters in the transition modeling are compared to those of the more robust Simplex Stochastic Collocation\textsuperscript{16,17} (SSC) method.

The paper is organized as follows. Sections II and III give a brief overview of the flow solver and the models used in this work. Furthermore, in Section III an approach to quantify the uncertainties associated with the physical processes in boundary layer transition is proposed, which is particularly important for transonic flows. The SC and SSC methods employed for the uncertainty propagation are discussed in Section IV. After the validation for the flat plate in Section V, the uncertainty analysis of the VKI turbine vane is considered in Section VI and VII. The main conclusions are summarized in Section VIII.

II. Reynolds-averaged Navier-Stokes solver

The computations were performed using the in-house Reynolds-averaged Navier-Stokes (RANS) solver, developed at the Center for Turbulence Research at Stanford University. The flow solver is a parallel solver for the solution of the compressible Navier-Stokes equations on unstructured meshes based on a finite volume formulation and implicit time-integration on arbitrary polyhedral mesh elements.\textsuperscript{3,10}

The governing equations are written in conservative form as

$$\frac{\partial}{\partial t} \int_{\Omega} U d\Omega + \int_{\partial\Omega} [F(U) - F_v(U)] dA = 0,$$

where $U = U(x,t)$ is the state variable, $F(U)$ and $F_v(U)$ are the convective and viscous fluxes, respectively, and $\Omega$ and $\partial\Omega$ are the physical domain of interest and its boundary.

In particular, we consider

$$U = [\rho, \rho v, E]^T,$$

$$F(U) = [\rho v, \rho v^2 + p n, (E + p) (v \cdot n)]^T,$$

$$F(U)_v = [0, n \cdot \Pi, v \cdot (n \cdot \Pi) + n \cdot Q]^T,$$

where $\rho, v, p, E, \Pi, Q, n$ represent density, Cartesian velocity vector, pressure, total energy, stress tensor, heat flux vector and outward pointing unit vector normal to the surface, respectively. The code is entirely written in C++ and uses subdomain decomposition and the message passing interface (MPI) as the parallel infrastructure.

The flow quantities are integrated in conservative form using a Newton-Rhapson implicit scheme:

$$\left( \frac{I}{\Delta t} + \frac{\partial R}{\partial U} \right) \Delta U = -R(U^n), \text{ with: } R(U) = \frac{1}{V} \sum_{f} [F(U) - F_v(U)] A_f .$$

A Taylor expansion is used to formulate the Jacobian matrices $\partial R/\partial U$ for the inviscid and viscous fluxes. The inviscid fluxes are computed with the HLLC approximate Riemann solver\textsuperscript{3,15} based on state values reconstructed at the face center using a spatial second order approximation. The viscous fluxes are approximated with a central difference approximation, likewise leading to a second order accurate scheme. The resulting large sparse system (the Jacobian matrices for the inviscid fluxes are obtained using first-order discretization) is solved with the generalized minimal residual method (GMRES) using the freely available linear solver package PETSc.\textsuperscript{12}

The scalar transport equations for the turbulence and transition model are solved segregated after each pseudo time step for the Navier-Stokes equations. The transport equation for a generic scalar $\phi$ can be written in conservative form as

$$\frac{\partial}{\partial t} \int_{\Omega} (\phi) d\Omega + \int_{\partial\Omega} [\phi (n \cdot \rho v) - F_v(\phi)] dA = \int_{\Omega} S(\phi) d\Omega ,$$

where $S(\phi)$ is the scalar source term and $F_v(\phi)$ is the viscous flux for the scalar considered.
III. Laminar-turbulent transition modeling

In order to account for laminar to turbulent transition in the test cases considered, the $\gamma - \widetilde{Re}_\theta$ correlation based transition model of Langtry and Menter\textsuperscript{5} is used. It is a framework for implementing empirical transition criterions in general-purpose flow solvers that can be used in an unstructured and parallel code. The model solves two transport equations, namely the intermittency factor $\gamma$ that describes the fraction of time in which the flow at a certain location is turbulent, and the momentum thickness Reynolds number $\widetilde{Re}_\theta$ that is used to trigger the transition onset. The intermittency is one in the free stream and in the fully developed turbulent boundary layer and zero in the pre-transitional laminar boundary layer.

III.A. Transition model

The transport equations in differential form are given by

$$\frac{\partial \rho \gamma}{\partial t} + \frac{\partial \rho u_i \gamma}{\partial x_i} = P_\gamma - E_\gamma + \left[ \left( \mu + \frac{\mu_t}{\sigma_f} \right) \frac{\partial \gamma}{\partial x} \right],$$  

(5)

$$\frac{\partial \rho \widetilde{Re}_\theta}{\partial t} + \frac{\partial \rho u_i \widetilde{Re}_\theta}{\partial x_i} = P_{\theta t} + \left[ \sigma_{\theta t} (\mu + \mu_t) \frac{\partial \widetilde{Re}_\theta}{\partial x} \right],$$  

(6)

where $P_\gamma$ is the production and $E_\gamma$ is the destruction of the intermittency, respectively. The term $P_{\theta t}$ in (6) is the production of the momentum thickness Reynolds number and is given by

$$P_{\theta t} = c_{\theta t} \frac{\rho}{I} \left( Re_{\theta t} - \widetilde{Re}_{\theta t} \right) \left(1 - F_{\theta t}\right),$$  

(7)

with the empirical correlation criterion for the transition onset $Re_{\theta t}$.

The model is applicable to incompressible transitional flows. In transonic flows, compressibility and the presence of shock waves affect the onset and extend of transition considerably, and should be incorporated into the model. Unfortunately, there is only limited open literature about these compressibility effects on boundary layer transition. Qualitatively, increasing the Mach number results in a longer transition zone and a delay of the onset. However, a large discrepancy among different correlations, proposed by various groups, exists when this effect is quantified. These studies investigated the Mach number influence on the growth rate of the turbulent spots in the transitional boundary layer and the various results showed a decrease of the spot growth rate at Mach=1 by a factor of 1.16 to almost 6.\textsuperscript{14} Similar discrepancies exist regarding the Mach number influence on the transition onset.

Instead of developing a correlation for the compressibility effect to match a particular experimental data set, this work focuses on quantifying the above mentioned uncertainties. Initially, given correlations are used and then utilized to assess the lack of knowledge, respectively the large uncertainties associated with the compressibility. Effectively, the decrease of the turbulent spot growth with increasing Mach number can be incorporated into the $\gamma - \widetilde{Re}_\theta$ model by changing the intermittency production term $P_\gamma$. Steelant and Dick\textsuperscript{14} found that the intermittency production $P_\gamma$ is proportional to the square root of the turbulent spot growth parameter, hence $P_\gamma$ can be modified by

$$P_\gamma^* = \sqrt{f(Ma_{is})} P_\gamma$$  

(8)

with $f(Ma_{is})$ as a function of the isentropic Mach number $Ma_{is}$. A simple linear function is constructed and given by:

$$f(Ma_{is}) = 1 - (0.14 + 0.7U_P) Ma_{is},$$  

(9)

with $U_P$, a uniformly distributed random variable on the domain $[0; 1]$ to assess the uncertainty of the spot growth parameter mentioned above. For $Ma_{is} = 1$ a decrease of 16% to 86% can be achieved and for $Ma_{is} = 0$ the function returns 1 and therefore does not change $P_\gamma$ for incompressible flows. Similarly, the transition onset delay for transonic flows is incorporated by changing $Re_{\theta t}$ in (7) as a function of $Ma_{is}$:

$$Re_{\theta t}^* = (1 + 0.6U_{Re_{\theta t}} Ma_{is}) Re_{\theta t},$$  

(10)

with $U_{Re_{\theta t}}$ a uniformly distributed random variable on the domain $[0; 1]$. For $Ma_{is} = 0$ the transition onset criterion $Re_{\theta t}$ returns its incompressible value, whereas for $Ma_{is} = 1$ a maximum increase of 60% can be achieved.
III.B. Underlying turbulence model

The $\tilde{\gamma} - \tilde{Re}_{\theta t}$ transport equations use the $k-\omega$ shear stress transport (SST) turbulence model\(^7\) as underlying model to provide the turbulent quantities $k-\omega$ and the eddy viscosity $\mu_t$ for the Navier-Stokes solver. As the SST model has gained an increasing popularity during the last years a variety of modifications to various limiters within the model have been introduced; hence, different versions are available. The most important limiter functions of the model version used throughout this work are given below. The limiter for the eddy viscosity is given by:

$$\mu_t = \rho a_1 k \left/ \max (a_1 \omega, \Omega F_2) \right.,$$

which uses the magnitude of the vorticity $\Omega$ in the denominator of the limiter. The production limiter, that ensures realizable turbulent Reynolds stress tensors is:

$$P_k = \min (P_k, 20C_\mu k \omega).$$

IV. Uncertainty quantification methods

IV.A. Stochastic Collocation

The Stochastic Collocation (SC) method\(^2,18\) is used to compute the effect of the input and model uncertainties on the laminar-turbulent transition in the turbomachinery application. The main idea behind Stochastic Collocation is to sample the quantity of interest at particular points in the parameter space. Integral statistics such as mean and standard deviation are then computed using quadrature or cubature rules. Non–integral statistics in the form of probability density functions are obtained from the Monte Carlo samples of the Lagrange interpolation of the quantity of interest over the sample points. In the case of a one–dimensional parameter space, these points are quadrature abscissas selected according to the probability measure of the random variable. Tensor product and sparse grid constructions are then utilized to generate the multi–dimensional abscissas.

The choice of one–dimensional abscissas defines the properties of the interpolation formula or the integration rule. Common choices of these nodes are the Clenshaw–Curtis (CC) and Gaussian abscissas. The Clenshaw–Curtis abscissas are the extrema of the Chebyshev polynomials in the interval $[-1,1]$, see Fig. 1 for a two–dimensional example. For any choice of $n_p > 1$, these points are given by

$$y^j = -\cos \left( \frac{\pi (j - 1)}{n_p - 1} \right), \quad j = 1, \ldots, n_p,$$

which renders to a nested rule in the sense that the set of lower order quadratures abscissas for $n_p = 2^i + 1$ is a subset of that of a higher order one with $n_p = 2^j + 1$ for integer values $i < j$. We adopt the CC rule in this work, since this hierarchical sampling property allows for estimating the quadrature error by comparing the results of different collocation orders. The response statistics are then computed as follows. Let $f(y_1, \ldots, y_d)$
be the quantity of interest as a function of independent random variables \( y_1, \ldots, y_d \) each with a probability density function \( \rho_k : \Omega \to \Gamma_k \subset \mathbb{R} \), \( k = 1, \ldots, d \), and let
\[
\Lambda := \{y_1^{n_1}, \ldots, y_1^{n_{n_1}}\} \times \cdots \times \{y_d^{n_1}, \ldots, y_d^{n_{n_d}}\},
\]
denote the set of multi-dimensional abscissas constructed by taking the tensor product of one-dimensional abscissas corresponding to each random variable \( y_k \). The integral- and statistics of \( w \) where 
\[
E[g(f)] = \int_{\Gamma_1} \cdots \int_{\Gamma_d} g(f)(y_1, \ldots, y_d) \prod_{k=1}^d \rho_k(y_k) dy_1 \cdots dy_d
\]
\[
\approx \sum_{j_1=1}^{n_{n_1}} \cdots \sum_{j_d=1}^{n_{n_d}} g(f)(y_1^{j_1}, \ldots, y_d^{j_d}) w_1^{j_1} \cdots w_d^{j_d},
\]
where \( w_k^{j_k} \) is the weight associated with the quadrature point \( y_k^{j_k} \) and \( g \) is a function defining the desired statistics of \( f \). Clearly, the accuracy of the approximation (15) depends on the location \( y_k^{j_k} \) of each absciss as well as the total number of points \( n_{n_k} \) along each direction \( k \). Notice that cardinality of \( \Lambda \) increases exponentially fast with respect to the number of random variables \( d \) and \( n_{n_k} \). The tensor product collocation scheme is therefore not computationally efficient for systems with large number of random inputs.

Alternatively, one can first construct an interpolant of \( f \) from samples in \( \Lambda \) and then estimate the integral and non-integral statistics of \( f \) using Monte Carlo samples of the interpolant. For this purpose, typically, a multi-dimensional Lagrange interpolation is adopted. More specifically, let
\[
l_j^k(y_k) = \prod_{i_k=1, i_k \neq j_k}^{n_{n_k}} \frac{y_k - y_k^{i_k}}{y_k^{j_k} - y_k^{i_k}},
\]
where \( n_{n_k} \) is the number of each abscissas \( \Lambda_{n_k} \) and \( \Lambda_{n_k} \) the one-dimensional Lagrange polynomials corresponding to abscissa \( y_k^{j_k} \), then the interpolant \( I(f) \) of \( f \) is given by
\[
I(f)(y_1, \ldots, y_d) = \sum_{j_1=1}^{n_{n_1}} \cdots \sum_{j_d=1}^{n_{n_d}} f(y_1^{j_1}, \ldots, y_d^{j_d})(l_1^{j_1} \otimes \cdots \otimes l_d^{j_d})(y_1, \ldots, y_d),
\]
where \( \otimes \) denotes the Kronecker product. The desired statistics are approximated from samples of \( I(f) \), for example, the expected value is computed as
\[
E[g(f)] \approx \frac{1}{N} \sum_{i=1}^N g(f)(y_1^i, \ldots, y_d^i),
\]
where \( y_1^i, \ldots, y_d^i, i = 1, \ldots, N \), are independent and identically distributed samples of \( y_1, \ldots, y_d \).

### IV.B. Simplex Stochastic Collocation

The global polynomial approximation of Stochastic Collocation can, however, be unreliable in case of large gradients in the response surface. This situation can, for example, occur for the heat flux in the transition region when the transition location is impacted by uncertainty. Also the spectral convergence of the Stochastic Collocation method can significantly reduce with an increasing number of uncertainties, due to the structured grid of the quadrature points in multiple random dimensions.

Here, the Simplex Stochastic Collocation (SSC) method\(^{16,17} \) is presented that combines the accuracy of polynomial interpolation with the robustness and the effectiveness in higher dimensions of Monte Carlo simulation based on random sampling. SSC discretizes the parameter space \( \Xi \) using non-overlapping simplex elements \( \Xi_j \) from a Delaunay triangulation of sampling points, with \( \Xi = \bigcup_{j=1}^{n_e} \Xi_j \) and \( n_e \) the number of elements. In each of the simplexes \( \Xi_j \), the response surface of the quantity of interest \( u(\xi) \) as function of the random parameters \( \xi \in \Xi \) is approximated by a polynomial \( w_j(\xi) \)
\[
w_j(\xi) = \sum_{m=0}^{P} c_{j,m} \Psi_{j,m}(\xi),
\]
with \( P + 1 \) coefficients \( c_{j,m} \) and basis polynomials \( \Psi_{j,m}(\xi) \). The polynomials are found by the interpolation of the samples \( v_{k} = u(\xi_{k}) \) at the vertexes \( \xi_{k} \) of the simplex elements, with \( k = 1, \ldots, n_{s} \) and \( n_{s} \) the number of samples. For higher degree interpolation, a stencil of sampling points \( v_{k,j,l} \) in the vertexes \( \xi_{k,j,l} \) of surrounding simplexes is constructed, with \( l = 0, \ldots, N \) and \( k,j,l \in \{1, \ldots, n_{s}\} \). The polynomial coefficients \( c_{j,m} \) are then given by

\[
\begin{bmatrix}
    \Psi_{j,0}(\xi_{k,0}) & \Psi_{j,1}(\xi_{k,0}) & \cdots & \Psi_{j,P}(\xi_{k,0}) \\
    \Psi_{j,0}(\xi_{k,1}) & \Psi_{j,1}(\xi_{k,1}) & \cdots & \Psi_{j,P}(\xi_{k,1}) \\
    \vdots & \vdots & \ddots & \vdots \\
    \Psi_{j,0}(\xi_{k,N}) & \Psi_{j,1}(\xi_{k,N}) & \cdots & \Psi_{j,P}(\xi_{k,N}) \\
\end{bmatrix}
\begin{bmatrix}
    c_{j,0} \\
    c_{j,1} \\
    \vdots \\
    c_{j,P} \\
\end{bmatrix}
= \begin{bmatrix}
    v_{k,j,0} \\
    v_{k,j,1} \\
    \vdots \\
    v_{k,j,N} \\
\end{bmatrix},
\]

with \( N \geq P \). The robustness of the approximation is guaranteed by using a limiter approach for the local polynomial degree \( p_{j} \), based on the extension of the Local Extremum Diminishing (LED) concept to probability space. This ensures that no overshoots are present in the response interpolation in each of the elements \( \Xi_{j} \)

\[
\min_{\Xi_{j}}(w_{j}(\xi)) \geq \min_{\Xi_{j}}(u(\xi)) \land \max_{\Xi_{j}}(w_{j}(\xi)) \leq \max_{\Xi_{j}}(u(\xi)),
\]

for \( j = 1, \ldots, n_{e} \). The initial samples consist of the extrema of the parameter ranges and one at the nominal conditions, see Fig. 2(a) for a two-dimensional example. The discretization is adaptively refined by calculating a refinement measure based on a local error estimate in each of the simplex elements. A new sampling point is then added randomly in the simplex with the highest measure and the Delaunay triangulation is updated. The sample is confined to a subdomain of the simplex to ensure a good spread of the sampling points, see Fig. 2(a). The refinement to \( n_{s} = 17 \) samples, shown in Figure 2(b), leads to a superlinear convergence by increasing the polynomial degree \( p_{j} \) with the increasing number of available samples \( n_{s} \). The sampling procedure is stopped when a global error estimate reaches an accuracy threshold.

V. Model validation for incompressible flows

The \( \gamma - \tilde{R}_{e_{gl}} \) model implementation is first validated for transitional flows over adiabatic flat plate test cases\(^5\) that have been used to calibrate the model coefficients and that are commonly used as benchmarks for transition models. The examples study the influence of the Reynolds number, the inlet turbulence intensity, and the pressure gradient on the boundary layer transition along the plate.\(^13\) In particular, three cases were considered in this work denoted as T3A, with a zero pressure gradient boundary layer, and T3C3 and T3C4 both with a pressure gradient similar to an aft loaded turbine blade. The computational domain is modeled with H-type grids with \( y^{+} \) values below 0.3 in the first layer above the walls. Fig. 3 shows the results for the skin friction distribution \( c_{f} \) and the normalized turbulent kinetic energy \( k/U_{0}^{2} \) in the free stream with respect to \( Re_{x} \) for the cases T3A and T3C4. The same results are obtained compared to the results in literature,\(^5\) which validates the correct model implementation for these test cases.

In order to account for the measurement uncertainty of the turbulence inlet conditions, the inlet turbulence intensity \( T_{u_{in}} \) and the turbulence Reynolds number \( Re_{T} \) are considered to be uncertain with independent uniform distributions; for the test case T3C3 we assume \( T_{u_{in}}=[4.86; 5.94] \) and \( Re_{T}=[72; 88] \), respectively. SC results are shown in Fig. 4(a) for the mean and the 95% uncertainty interval of the free
stream turbulent kinetic energy $k/U_0^2$ along the plate, compared to the deterministic simulation and experimental data.\footnote{13} The decaying turbulent kinetic energy with $Re_x$ results in a decreasing uncertainty interval from the uncertain inlet condition for $Tu_{in}$. This behavior is confirmed by the decreasing standard deviation in Fig. 4(b). The deterministic result is close to the mean value and in agreement with the experiments. The accuracy of the UQ approximations of Fig. 4 is shown in Fig. 5. The quadrature error in the approximation of the mean and standard deviation for a CC rule with $n_p$ points is estimated by the difference with the results for $(n_p - 2)$ CC points. This implies for $n_p = 3$ that the mean is compared to the deterministic value for $n_p = 1$. The error estimate has a maximum near the start of the plate at $Re_x = 0$ of 0.61\% for the standard deviation.

![Image](image1.png)

(a) Case T3A with zero pressure gradient  
(b) Case T3C4 with pressure gradient and separation induced transition

Figure 3. Assessment of the correct model implementation for the flat plate test cases T3A and T3C4.

The uncertainty in $Tu_{in}$ and $Re_T$ has a different effect on the skin friction $c_f$ as function of $Re_x$ of Fig. 6. The mean and the 95\% interval in Fig. 6(a) show that $Tu_{in}$ and $Re_T$ have a small effect on the transition location. The standard deviation has a maximum in Fig. 6(b) in the transition region. The errors are also largest in the transition region. The accuracy of the mean approximation improves significantly from $n_p = 3$ to $n_p = 5$. The error in the standard deviation is with a maximum of 10.2\% at the transition point also of an acceptable magnitude.

VI. Boundary layer transition on a transonic turbine guide vane

The turbine guide vane considered in this work has been experimentally investigated by Arts et al.\footnote{1} in order to study the influence of Mach number, turbulence intensity, and Reynolds number on the transitional heat transfer distribution $h = \dot{q}_w/(T_\infty - T_w)$. The geometry of the blade is defined by the chord=67.647mm, pitch to chord ratio=0.85, throat to chord ratio=0.2207, and a stagger angle of 55 deg measured from the axial direction. The total inlet temperature is set at $T_0=420K$ and the wall temperature is considered to

![Image](image2.png)

(a) Mean and 95\% interval  
(b) Standard deviation

Figure 4. Effect of uncertainty on the free stream turbulent kinetic energy (case T3C3).
Figure 5. Quadrature error in the mean and standard deviation approximation for the free stream turbulent kinetic energy using SC.

Figure 6. Effect of uncertainty on the skin friction coefficient $c_f$ (case T3C3).

Figure 7. Quadrature error in the mean and standard deviation approximation for the skin friction coefficient $c_f$ using SC.
be close to a constant value of 300K in the experiments. The flow is transonic with an exit Mach number close to one. The incoming turbulence level $T_u$ was measured 55mm upstream of the leading edge plane. The uncertainty on the heat transfer measurement was of the order of ±5%. As no turbulent length scale or dissipation was measured the turbulence Reynolds number $Re_T$ is considered to be uncertain within a reasonable range, estimated using the correlation of Roach. The computational domain consists of 25,000 control volumes and the $y^+$ values are less than one for all cells at the wall surface.

VI.A. Initial accuracy assessment of SC for the MUR241 test case

Preliminary SC results, primarily to assess the accuracy of this collocation method, are given for the test case MUR241 with a Reynolds number (based on the chord length and exit conditions) of $Re = 2.1\times10^6$ and $T_u = 6\%$ and MUR235 with $Re = 1.1\times10^6$ and $T_u = 6\%$. The stochastic results for the surface heat transfer are given in Fig. 8 for the MUR241 case in comparison with experimental data as function of the curvilinear coordinate along the blade s/c normalized by the chord. The reported 5% uncertainty in the experimental results is indicated by the uncertainty bars. Positive values of s/c denote the suction side of the airfoil and the pressure side is parametrized by negative s/c-values. The experiment indicates transition on the suction side at s/c = 0.6, while on the pressure side no clear transition location can be identified due to the smooth increase of the heat transfer towards the trailing edge. The uncertainty in $T_u$ and $Re_T$ in the ranges [5.7; 6.3] and [50; 150] has a significant effect on the transition location on the suction side in Fig. 8(a). Consequently the 95% uncertainty interval in the transition region at s/c = 0.6 shows a large range of [130; 1370.2], which increases with $n_p$. The same divergence of the standard deviation with increasing $n_p$ at s/c = 0.6 can be seen in Fig. 8(b). The large overshoot in computed heat transfer at the rear part of the suction side needs to be accounted for by model form uncertainty in addition to the varying turbulence inflow conditions.

![Figure 8. Heat transfer for the VKI test case MUR241.](image)

When the 95% uncertainty interval is compared to the CC samples in Fig. 9, it is observed that the interval exceeds all the samples at s/c = 0.6. This is not to be expected since the CC points include the extreme values of the uncertain input ranges [5.7; 6.3] and [50; 150]. In Fig. 10 it is shown that these diverging overshoots are caused by the oscillatory global polynomial SC interpolation of the CC points of the discontinuous response surface. These oscillations increase with $n_p$ such that the results do not convergence for $n_p = 5$ as shown for the cumulative distribution function (CDF) at s/c = 0.6 in Fig. 11(a). This leads to an 18.1% error in the local standard deviation at s/c = 0.6 for $n_p = 5$ in Fig. 11(b). It can also be concluded from Fig. 10 that the uncertainty in the turbulence intensity $T_u$ has no significant effect on the heat transfer. In the following, the aleatoric uncertainty in $Re_T$ is, therefore, combined with epistemic uncertainty in the laminar-turbulent transition model, while the inlet turbulence intensity $T_u$ is kept constant using its nominal value.
Figure 9. Samples and 95% interval of the heat transfer for the VKI test case MUR241.

Figure 10. Samples and response surface approximation of the heat transfer for the VKI blade at s/c = 0.6.

Figure 11. Cumulative distribution function (CDF) and quadrature error in the mean and standard deviation approximation.
VI.B. Initial accuracy assessment of SC for the MUR235 test case

For the MUR235 case of the VKI turbine guide vane the turbulence Reynolds number $Re_T$ is uniformly distributed in $[600; 900]$ and uncertainty is injected in Eq. (10) that determines the onset of transition. No transition is observed in the experiment along the pressure side, while a sharp increase of the heat transfer at $s/c = 0.8$ clearly indicates transition at the suction side. The epistemic uncertainty in the transition model accounts to a large extent for the difference between the deterministic simulation and the experiments in Fig. 12. No uncertainty has been added for the spot growth parameter in this preliminary result. It is clearly visible in Fig. 12(b) that the uncertain transition model injects standard deviation in the transition region. This also significantly affects the prediction of the mean heat transfer at $s/c = 0.6$ in the transition region, from the deterministic value of 298.4 to 373.6 and 417.8 for $n_p = 3$ and $n_p = 5$. The increasing mean value with respect to the deterministic value of 298.4 can also be deduced from the approximation of the CDF in Fig. 13(a). The error estimate for the standard deviation approximation with $n_p = 5$ indicates a relative error of 12.7% at $s/c = 0.6$, see Fig. 13(b).

![Figure 12. Heat transfer for the VKI test case MUR235.](image)

![Figure 13. Cumulative distribution function (CDF) and quadrature error in the mean and standard deviation of the heat transfer for the VKI test case MUR235.](image)

VII. Results for the VKI MUR 235 and MUR 241 cases using SC and SSC

Because of the diverging results for Stochastic Collocation (SC) in the transition region the convergence is computed up to more quadrature points and the results are compared to those of Simplex Stochastic Collocation (SSC) hereafter. Since the epistemic uncertainty in the transition model had significant impact on the results for the transonic turbine guide vane, also a second source of epistemic uncertainty is considered.
in terms of the turbulent spot growth parameter as described in section III. Therefore, three uncertainties are considered: the inlet turbulent length-scale given by $Re_T$, the correlation that determines the transition onset location and the correlation for the turbulent spot growth within the transitional boundary layer.

VII.A. MUR241 test case

In this case the turbulence Reynolds number $Re_T$ is uniformly distributed in $[140; 260]$. For SC, three levels of Clenshaw–Curtis rules are considered with $n_p = \{3, 5, 9\}$, which leads for three uncertainties to $n = \{27, 125, 729\}$ samples. The results of SSC are shown for $n = \{10, 50, 100\}$ samples.

Fig. 14 shows that the turbulent spot growth parameter has a significant impact on the heat transfer downstream of the transition point on the suction side of the blade in terms of the mean and the 99% interval. The epistemic uncertainty in the turbulent spot growth modeling, therefore, gives a good explanation for the discrepancy between the simulation and the experiment downstream of the transition point in Fig. 8. It leads for SC in overshoots in the transition region and results that do not show convergence for up to $n = 729$ samples. The results of SSC are better behaved with good agreement between the prediction with 50 and 100 samples. Similar observations can be made from the comparison of the results for the standard deviation in Fig. 15.

![Figure 14. Mean and 95% interval of the heat transfer for the VKI test case MUR241.](image)

![Figure 15. Standard deviation of the heat transfer for the VKI test case MUR241.](image)

VII.B. MUR235 test case

For the case MUR235 $Re_T$ has been estimated to be uniformly distributed between $[490; 910]$. The additional epistemic uncertainty in the turbulent spot growth has a smaller effect on the results in Fig. 16 for $Re_T$ in
[490:910] compared to MUR241 presented above. However, it results in a larger spread of the SC predictions, while SSC shows converged results at $n = 100$ for the mean and the uncertainty interval. The standard deviation of the heat transfer in Fig. 17 confirms these observations.

Figure 16. Mean and 95% interval of the heat transfer for the VKI test case MUR235.

Figure 17. Standard deviation of the heat transfer for the VKI test case MUR235.

VIII. Conclusions

The ability to accurately predict the transition process is crucial for the design of efficient and reliable transonic gas turbine compressors. The boundary layer development, losses, efficiency, and heat transfer are known to be sensitive to small physical variations and uncertainties in the transition model. In this work, the impact of these epistemic uncertainties as well as the aleatory uncertainties is, therefore, quantified in the numerical uncertainty analysis of a transonic turbine guide vane.

The transition model implementation is first validated for the skin friction distribution $c_f$ and the normalized turbulent kinetic energy $k/\bar{U}_0^2$ in the free stream in the transitional flow over adiabatic flat plate benchmarks T3A and T3C4. The uncertainty analysis results for $c_f$ show that uncertainty in the inlet turbulence intensity $T_u$ and the turbulence Reynolds number $Re_T$ have a small effect on the location of the transition point. In the turbine guide vane test case MUR241, uncertainty in $Re_T$ has a significant effect on the transition location on the suction side. The impact of the variation in the turbulence intensity $T_u$ is negligible for this case. The aleatoric uncertainty in $Re_T$ and the epistemic uncertainty in the laminar-turbulent transition model for the MUR235 case shows that the model uncertainty accounts to a large extent for the difference between the deterministic simulation and the experiments. The uncertain transition model injects uncertainty in the transition region with a significant effect on the mean and standard deviation of
the surface heat transfer.

The additional epistemic uncertainty in the transonic spot growth parameter explains the discrepancy between the simulations and the experiments in the heat transfer downstream of the transition on the suction side of the blade for the MUR241 case. This problem with three uncertainties in $Re_T$ (inlet turbulent length-scale) and epistemic parameters $U_{Re_\theta}$ (transition onset correlation) and $U_P$ (turbulent spot growth) gives overshoots and results that are not yet converged for Stochastic Collocation with $n = 729$ samples. The comparison with Simplex Stochastic Collocation demonstrates the more robust convergence for $n = 100$ samples of the latter method.

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