A simplex-simplex approach for mixed aleatory-epistemic uncertainty quantification

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The Simplex Stochastic Collocation (SSC) method has been recently proposed in order to handle aleatory uncertainty quantification (UQ), i.e. irreducible variabilities inherent in nature. In this work, we present an extension of this method for treating epistemic uncertainty in the context of interval analysis approach. This numerical method is based on a Simplex space representation, high-order polynomial interpolation and adaptive refinements, permitting to treat mixed aleatory-epistemic uncertainties. This method displays good properties in terms of accuracy and computational cost. Several numerical examples are presented to demonstrate the properties of the proposed method.

I. Introduction

In most engineering applications, it is of great interest to consider physical and modeling uncertainties in computational mechanics field. Uncertainties can be aleatory, which are irreducible variabilities inherent in nature, or epistemic, which are reducible uncertainties resulting from a lack of knowledge. In this case, experimental measures are often too scarce for estimating statistic properties, such as probability distributions. Therefore, non-probabilistic method based on interval specifications or on a-priori Bayesian-type method should be applied. Generally, dealing with input uncertainties for realistic physical problems consist in treating mixed epistemic/aleatory uncertainties.

Several methods have been proposed for the computation of aleatory uncertainty (see Ref. 1 for a detailed review). For example, the stochastic methods based on polynomial chaos demonstrated their efficiency and are widely used (see Refs. 2–4). Another class of method for the UQ is based on the stochastic collocation (SC) approach, 5 where building interpolating polynomials are used in order to approximate the solution. Recently, Witteveen & Iaccarino proposed a simplex stochastic collocation 6–8 based on simplex elements, that can efficiently discretize non-hypercube probability spaces. It combines the Delaunay triangulation of randomized sampling at adaptive element refinements with polynomial extrapolation to the boundaries of the probability domain. This method achieves superlinear convergence and a linear increase of the initial number of samples with increasing dimensionality.

Tough a wide and large campaign to explore new methods for quantifying aleatory uncertainty, the numerical study of epistemic or mixed aleatory-epistemic uncertainty is much more challenging basing on the complexity of the mathematical formulation. Several methods for characterizing and modeling epistemic uncertainty exist in literature, including possibility theory, 9 fuzzy set theory, Dempster-Shafer evidence theory, 10 second-order probability. 11 In 2010, Jakeman et al. 12 proposed a numerical treatment of epistemic uncertainty based on solving an encapsulation problem, without using any probability information, in a hypercube that encapsulates the unknown epistemic probability spaces.

Another interesting approach deals with interval analysis. 13, 14 Interval analysis can be considered a simpler approach with respect to the other methods, since computational problem is converted in an optimization problem, where interval on the outputs should be found by starting from a given inputs defined within intervals. Even if simple, the choice of an optimization strategy permitting to reduce the computational cost by preserving accuracy is not straightforward. A direct approach is to use optimization to find the maximum and minimum values of the output measure of

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Then, randomness can be expressed as \( \xi \) variables, statistical analysis based on aleatory uncertainties is performed, thus computing the associated statistical one with the same probability of occurrence. The aim of this analysis is to determine interval bounds on the output of moment of interest. By several sampling of epistemic variables, an ensemble of statistical moments is produced, each and boundary conditions in terms of \( \Omega \), the mixed aleatory/epistemic uncertainty framework reduces to solve the following problems

Consider the following computational problem for an output of interest, which correspond to the upper and lower interval bounds on the output. In Eldred et al.,\textsuperscript{14} they show that the coupling of local gradient-based and global nongradient-based optimizers with non-intrusive polynomial chaos and stochastic collocation expansion methods is highly effective, displaying a strong reduction of global number of simulations with respect to more classical approach.

In this work, we present an extension of the Simplex Stochastic Collocation method\textsuperscript{7,8} for treating epistemic uncertainty in the framework of interval analysis approach. In particular, this method consists in a multi-scale strategy based on simplex space representation in order to minimize global cost of mixed epistemic-aleatory uncertainty quantification. This reduction is obtained i) by a coupled stopping criterion, ii) by an adaptive polynomial interpolation that could be used as a response surface in order to accelerate optimization convergence, iii) by a simultaneous min/max optimization sharing the same interpolating polynomials at each iteration.

In section II, the computational problem and the numerical methods based on simplex representation and Nelder-Mead algorithm is presented. Moreover, another strategy for interval analysis, based on a Non-intrusive Polynomial Chaos and a Genetic Algorithm method is presented and used as term of comparison for the proposed strategy based on Simplex. In section IV, several results are presented and the efficiency of the proposed approach demonstrated. Finally, in section V, conclusions and perspectives are drawn.

II. Methodology

II.A. Problem Definition

Consider the following computational problem for an output of interest \( u(x, t, \xi(\omega)) \)

\[
\mathcal{L}(x, t, \xi(\omega); u(x, t, \xi(\omega))) = S(x, t, \xi(\omega)),
\]

with appropriate initial and boundary conditions.

The operator \( \mathcal{L} \) and source term \( S \) are defined on domain \( D \times T \times \Xi \), where \( x \in D \) and \( t \in T \) are the spatial and temporal coordinates with \( D \subset \mathbb{R}^d \), \( d \in \{1, 2, 3\} \), and \( T \subset \mathbb{R} \). Randomness is introduced in (1) and its initial and boundary conditions in terms of \( n_L \) second-order random parameters \( \xi(\omega) = \{\xi_1(\omega_1), \ldots, \xi_n_L(\omega_{n_L})\} \in \Xi \) with parameter space \( \Xi \subset \mathbb{R}^{n_L} \). The symbol \( \omega = \{\omega_1, \ldots, \omega_{n_L}\} \in \Omega \subset \mathbb{R}^{n_L} \) denotes events in the complete probability space \((\Omega, \mathcal{F}, P)\) with \( \mathcal{F} \subset 2^\Omega \) the \( \sigma \)-algebra of subsets of \( \Omega \) and \( P \) a probability measure. The random variables \( \omega \) are by definition standard uniformly distributed as \( \mathcal{U}(0, 1) \). Random parameters \( \xi(\omega) \) can have any arbitrary probability density \( f_L(\xi(\omega)) \). The argument \( \omega \) is dropped here on to simplify the notation. The objective of uncertainty propagation is to find the probability distribution of \( u(x, t, \xi) \) and its statistical moments \( \mu_{u_i}(x, t) \) given by

\[
\mu_{u_i}(x, y, t) = \int_{\Xi} u(x, t, \xi) f_L(\xi) d\xi,
\]

In interval analysis, aleatory and epistemic uncertainties are taken into account separately. For each set of epistemic variables, statistical analysis based on aleatory uncertainties is performed, thus computing the associated statistical moment of interest. By several sampling of epistemic variables, an ensemble of statistical moments is produced, each one with the same probability of occurrence. The aim of this analysis is to determine interval bounds on the output of interest in the case of mixed aleatory-epistemic uncertainties.

Let us suppose to have \( n_{L1} \) epistemic random variables and \( n_{L2} \) aleatory random variables, where \( n_{L1} + n_{L2} = n_L \). Then, randomness can be expressed as \( \xi_1(\omega) = \{\xi_1(\omega_1), \ldots, \xi_{n_{L1}}(\omega_{n_{L1}})\} \in \Xi_1 \) with parameter space \( \Xi_1 \subset \mathbb{R}^{n_{L1}} \) and \( \xi_2(\omega) = \{\xi_{n_{L1}+1}(\omega_{n_{L1}+1}), \ldots, \xi_{n_L}(\omega_{n_L})\} \in \Xi_2 \) with parameter space \( \Xi_2 \subset \mathbb{R}^{n_{L2}} \). Using interval analysis in a mixed aleatory/epistemic uncertainty framework reduces to solve the following problems

\[
\min_{\xi_1(\omega) \subset \mathbb{R}^{n_{L1}}} \mu_{u_i}(x, t) \quad \text{and} \quad \max_{\xi_1(\omega) \subset \mathbb{R}^{n_{L1}}} \mu'_{u_i}(x, t)
\]

with

\[
\mu_{u_i}(x, t) = \int_{\Xi_2} u(x, t, \xi_2) f_L(\xi_2) d\xi_2,
\]

where \( \mu_{u_i}(x, t) \) is the statistical quantity of interest computed with respect to the \( n_{L2} \) aleatory uncertainty. Solving problem expressed in 3 gives the interval bounds on the output of interest.

Though the generality of this formulation, in this work, the reliability indices,\textsuperscript{14,15} \( \beta_{CDF} \) and \( \beta_{CCDF} \), computed as
A local UQ method computes these weighted integrals over parameter space \( \Xi_2 \) as a summation of integrals over \( n_e \) disjoint subdomains \( \Xi_2 = \bigcup_{j=1}^{n_e} \Xi_j \)

\[
\mu_{ui}(x, t) = \sum_{j=1}^{n_e} \int_{\Xi_j} u(x, t, \xi_2)^{\top} f_{L}(\xi_2) \, d\xi_2.
\]  

(6)

whereas in SSC the integrals in the simplex elements \( \Xi_j \) are computed by approximating response surface \( u(\xi_2) \) by an interpolation \( w(\xi_2) \) of \( n_s \) samples \( v = \{v_1, \ldots, v_{n_s}\} \).

Here the arguments \( x \) and \( t \) are omitted for clarity of the notation.

Non-intrusive SSC uncertainty quantification method \( g \) then consists of a sampling method \( g \) and an interpolation method \( h \), for which holds \( w(\xi_2) = g(u(\xi_2)) = h(g(u(\xi_2))) \).

The sampling method \( g \) selects the sampling points \( \xi_{2k} \) for \( k = 1, \ldots, n_s \) and returns the sampled values \( v = g(u(\xi_2)) \), with \( v_k = g_k(u(\xi_2)) = u(\xi_{2k}) \).

Sample \( v_k \) is computed by solving (1) for realization \( \xi_{2k} \) of the random parameter vector \( \xi_2 \)

\[
\mathcal{L}(x, t, \xi_{2k}; v_k(x, t)) = S(x, t, \xi_{2k}),
\]  

(7)

for \( k = 1, \ldots, n_s \).

The interpolation of the samples \( w(\xi_2) = h(v) \) consists of a piecewise polynomial function

\[
w(\xi_2) = w_j(\xi_2), \quad \text{for } \xi_2 \in \Xi_j,
\]  

(8)

with \( w_j(\xi_2) \) a polynomial interpolation of degree \( p \) of the samples \( v_j = \{v_{k_{j,0}}, \ldots, v_{k_{j,N}}\} \) at the sampling points \( \{\xi_{2k_{j,0}}, \ldots, \xi_{2k_{j,N}}\} \) in element \( \Xi_j \), where \( k_{j,l} \in \{1, \ldots, n_s\} \) for \( j = 1, \ldots, n_e \) and \( l = 0, \ldots, N \), with \( N \) the number of samples in the simplexes.

The polynomial interpolation \( w_j(\xi_2) \) in element \( \Xi_j \) can then be expressed in terms of a truncated Polynomial Chaos expansion

\[
w_j(\xi_2) = \sum_{m=0}^{P} c_{j,m} \Psi_{j,m}(\xi_2).
\]  

(9)

where the polynomial coefficients \( c_{j,m} \) can be determined from the interpolation condition

\[
w_j(\xi_{2k_{j,l}}) = v_{k_{j,l}},
\]  

(10)

for \( l = 0, \ldots, N \), which leads to a matrix equation, that can be solved in a least-squares sense for \( N > P \).

The probability distribution function and the statistical moments \( \mu_{ui} \) of \( u(\xi_2) \) given by (6) are then approximated by the probability distribution and the moments \( \mu_{wi} \) of \( w(\xi_2) \)

\[
\mu_{ui}(x, t) \approx \mu_{wi}(x, t) = \sum_{j=1}^{n_e} \int_{\Xi_j} w_j(x, t, \xi_2)^{\top} f_{L}(\xi_2) \, d\xi_2,
\]  

(11)

in which the multi-dimensional integrals are evaluated using a weighted Monte Carlo integration of the response surface approximation \( w(\xi_2) \) with \( n_{mc} \gg n_e \) integration points.

This is a fast operation, since it only involves integration of piecewise polynomial function \( w(\xi) \) given by (8) and does not require additional evaluations of the exact response \( u(\xi_2) \).

For a complete description of the algorithm, see Refs. 7, 8
Simplex$^2$ Algorithm for epistemic uncertainty

Simplex$^2$ method is an efficient multi-scale coupling of the SSC, described in the previous section, and the Nelder-Mead (NM) algorithm. This method uses also a simplex space representation, constituted by $n_N + 1$ vertices. It generates new designs by extrapolating the behavior of the objective function measured at each one of the basic designs, that constitute the geometric simplex. The algorithm then chooses to replace one of these designs with the new one and so on. For all details concerning implementation of the basic NM method, we refer to classical reference.

Then, the Simplex$^2$ method is based on two different levels of the simplex. For each design variable, a stochastic simplex (micro-scale) is generated using SSC method. The set of design variables constitutes the so-called geometric simplex (macro-scale).

This method has four advantages. First one, a stopping criterion based on the minimal error in both the stochastic and geometric simplex can be defined. Second, it is possible to exploit interpolating polynomials one and so on. For all details concerning implementation of the basic NM method, we refer to classical reference.

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The Simplex$^2$ algorithm can be summarized as follows. The optimization space is named as $\Xi_{opt}$, and an element $j$ of $\Xi_{opt}$ as $\Xi_{opt,j}$. For each design point $y_0$, we define a probabilistic space $\Xi_0$. The notation $\Xi_{o,j}$ refers to an element $j$ of stochastic simplex associated to $y_o$. Let us suppose to maximize (minimize) a given function constituted by statistics of a given output, for example to maximize (minimize) mean

1. The initial grid of sampling designs $y_o$ is composed of the $2^n$ vertexes of the hypercube enclosing the optimization space $\Xi_{opt}$ and one sampling point in the interior. For each sampling design $y_o$, the following steps are followed.

   a. An initial grid of sampling points $\xi_k$ is composed of the $2^n$ vertexes of the hypercube enclosing the probability space $\Xi_0$ and one sampling point in the interior.

   b. The $n_{sinit}$ initial samples $v_k$ are computed by solving $n_{sinit}$ deterministic problems (1) for the parameter values corresponding to the initial sampling points $\xi_k$ located in $\Xi_0$ only.

   c. The initial discretization of parameter space $\Xi_0$ is constructed by making a Delaunay triangulation of all sampling points $\xi_k$ resulting in $n_v$ simplex elements $\Xi_j$.

   d. The polynomial approximation $w_j(\xi)$ in each of the interpolation elements $\Xi_j$ is constructed.

   e. The statistical moments $\mu_{1,j}$ of $w(\xi)$, and the error computed on this quantity (see Ref. 6 for more details) are calculated by Monte Carlo integration of (6). If the error is larger than the prescribed error, then a refinement on the stochastic simplex is performed. The criterion for the refinement is the following

   $\varepsilon_{SSC} = \mu_{1,j}(y_N) - \mu_{1,j}(y_1)$

2. For each sampling design $y_o$, $\mu_{1,j}$ is known.

3. The polynomial interpolation $w_j(y)$ (9) on $\Xi_{opt}$ is then constructed (denoted P1 in the following), by solving (9) on $\Xi_{opt}$.

4. Order the fitness function according to the values of $\mu_{1,j}$.

   $\mu_{1,j}(y_1) < \mu_{1,j}(y_2) \ldots < \mu_{1,j}(y_3)$

5. Compute the center of gravity $y_0$ of all points expect $y_{n+1}$.

6. Compute reflected point by means of the response surface P1, i.e. $y_r = y_0 + \alpha (y_0 - y_{n+1})$ with $\alpha = 1$. If the reflected point is better than the second worst, but not better than the best, i.e. $\mu_{1,j}(y_1) < \mu_{1,j}(y_r) \ldots < \mu_{1,j}(y_n)$, then obtain a new simplex by replacing the worst point $y_{n+1}$ with the reflected point $y_r$, and go from 1-a to 1-e in order to compute $\mu_{1,j}(y_r)$.  


7. If the reflected point is the point so far, \( \mu_{i+1}(y_r) < \mu_{i+1}(y_l) \), then compute by means of the response surface \( P_1 \), \textit{i.e.} the expanded point \( y_e = y_0 + \Gamma(y_0 - y_{n+1}) \) with \( \Gamma = 2 \). If the expanded point is better than the reflected point, \( \mu_{i+1}(y_e) < \mu_{i+1}(y_r) \), then obtain a new simplex by replacing the worst point \( y_{n+1} \) with the expanded point \( y_e \), and go from 1-a to 1-e compute exactly \( \mu_{i+1}(y_e) \). Else obtain a new simplex by replacing the worst point \( y_{n+1} \) with the reflected point \( y_r \), and go to step 6. Else (i.e. reflected point is not better than second worst) continue at step 9.

8. Here, it is certain that \( \mu_{i+1}(y_r) > \mu_{i+1}(y_n) \). Compute contracted point by means of the response surface \( P_1 \), \textit{i.e.} \( y_c = y_{n+1} + \rho(y_0 - y_{n+1}) \) with \( \rho = 0.5 \). If the contracted point is better than the worst point, \( i.e. \mu_{i+1}(y_c) < \mu_{i+1}(y_{n+1}) \) then obtain a new simplex by replacing the worst point \( x_{n+1} \) with the contracted point \( y_c \), and go from 1-a to 1-e compute exactly \( \mu_{i+1}(y_c) \). Else go to step 9.

9. For all but the best point, replace the point with \( y_i = y_1 + \sigma(y_i - y_1) \) with \( \sigma = 0.5 \). Go from 1-a to 1-e to compute exactly \( \mu_{i+1}(y_i) \).

10. Supposing a simultaneous optimization of the minimal and the maximal of the quantities of interest, the algorithm restarts from the step 3 of the algorithm by building a polynomial interpolation on the ensemble of points generated during both optimization.

### III.A. A reference strategy based on Polynomial Chaos and Genetic Algorithms

Another strategy for interval analysis is used as a basis for comparison with the Simplex\(^2\) strategy. This reference strategy, indicated in the following as PC-GA, is based on a non-intrusive Polynomial Chaos Method for the treatment of aleatory uncertainties and on a genetic algorithm (GA) for computing the bounds of the quantities of interest in the epistemic variables space. The coupling of these two techniques has been presented in Ref.\(^1\) GAs require evaluations of the fitness function for each individual in a generation, and this during several generations, until an optimal individual is selected: this is the major cause of their high computational cost. However, this drawback can be overcome if the fitness function is related to the design variables through an analytical expression. For this reason, the GA is coupled with an artificial neural network (ANN) for computing the bounds of the quantities of interest in the epistemic variables space (for more details concerning the ANN, see Refs.\(^{19,20}\)).

### IV. Results

#### IV.A. Rosenbrock stochastic problem

There exist several stochastic formulations of Rosenbrock optimization problem.\(^{21}\) In this work, we propose a slightly modified version, where a non-linear dependence on stochastic variable is taken into account. More precisely, we consider the following function

\[
 f(x) = \sum_{i=1}^{N-1} \left[ (1 - x_i)^2 + 100\sqrt{\varepsilon_i} + \alpha \left( x_{i+1} - x_i^2 \right)^2 \right]
\]  \tag{14}

where \( \varepsilon_i \) varies in \( Unif(0,1) \), \( x_i \) varies in \((-2;+2)\) and \( \alpha \) is taken equal to 1. This stochastic function has the same global optimum at \((1,1,1,...)\).

We show results obtained by comparing three different formulations. In the first one, called A1, SSC and NM methods have been used in a decoupled way, \textit{i.e.} NM method is used in its traditional version and SSC method is seen as a black-box. In the second formulation, called hereafter A2, estimated error on geometric simplex is used as stopping criterion for the error on stochastic simplex. Then, for design not close to optimum, the associated stochastic simplex will be less refined. In the third formulation (A3), polynomial extrapolation shown in 9 is applied to the space constituted by the epistemic uncertainties \((x_i)\). This allows reducing the global computational cost by estimating some transitory steps in Nelder-Mead algorithm without performing direct computations. Problem defined by (14) is considered with \( N = 2 \), then with two epistemic uncertainties and one aleatory uncertainty. First, we apply Simplex-Simplex method for minimizing \( \mu(f) \). In table 1, results obtained in terms of deterministic evaluations, where \( N_{al} \) is the number of iterations for NM algorithm and \( N_0 \) is the global number of deterministic evaluations, are reported for each formulation. Remark that a DOE constituted by 25 samples in the geometric simplex is used for each formulation, then with \( N_{al} = 25 \) and \( N_0 = 256 \), and then this computational cost is not reported in table 1. Exactly the same optimal design is obtained (the optimal fitness function is nearly \( 10^{-6} \) where the optimal theoretical fitness is equal to zero) in
both cases, without variations in $N_{it}$, but with a reduction of 40.8% for $N_0$ in the case A2. The same $N_{it}$ is obtained because convergence for $\mu(f)$ is more fast than convergence of the geometric simplex. The use of formulation A3 displays an important result in terms of computational cost, i.e. allowing a reduction of 66.3% for $N_0$ with respect to the decoupled formulation A1. In this case, 31 individuals are generated during the transitory steps in the Nelder-Mead algorithm (from step 6 to step 8) by using the response surface instead of a direct computation, that explains the strong saving achieved. Simplex evolution in the design variables space is reported in figure 1. For the optimal individual, the stochastic response surface with respect to the uncertainty is nearly coincident with the exact solution, as shown in figure 2. Then the same optimization is performed in order to minimize $\mu(f) + \sigma(f)$ (DOE constituted by 25 samples in the geometric simplex is used for each formulation, then with $N_{it} = 25$ and $N_0 = 274$), and the results in terms of $N_{it}$ and $N_0$ are reported in table 2. If A2 formulation is used, $N_0$ is reduced of 42.3%, with the same $N_{it}$. Results obtained by using A3 are impressive displaying a reduction of 66.3% for $N_0$. In fact, in this case, 29 individuals are estimated by using the response surface instead of direct computations.

<table>
<thead>
<tr>
<th>Formulation</th>
<th>$N_{it}$</th>
<th>$N_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>62</td>
<td>368</td>
</tr>
<tr>
<td>A2</td>
<td>62</td>
<td>218</td>
</tr>
<tr>
<td>A3</td>
<td>31</td>
<td>117</td>
</tr>
</tbody>
</table>

Table 1. Minimization of $\mu$ where $N_{it}$ is the number of iterations for NM algorithm, $N_0$ global number of deterministic evaluations

<table>
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</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>62</td>
<td>383</td>
</tr>
<tr>
<td>A2</td>
<td>62</td>
<td>221</td>
</tr>
<tr>
<td>A3</td>
<td>33</td>
<td>129</td>
</tr>
</tbody>
</table>

Table 2. Minimization of $\mu + \sigma$ where $N_{it}$ is the number of iterations for NM algorithm, $N_0$ global number of deterministic evaluations

Let us now focus on another version of the Rosenbrock problem. Here $\varepsilon_i + \alpha$ is assumed constant and equal to 1. Moreover, $x_1$ is considered an epistemic variable with an initial value -0.75 and bounds $-2 \leq x_1 \leq 2$, while $x_2$...
is taken as a normal random variable ($\mu = 0$ and $\sigma = 1$). The aim is to maximize $\beta_{CDF}$ for $z = 10$. The Simplex\textsuperscript{2} technique is successful in locating the optimum at the lower bound of $x_1$ with a computational cost of 305, 214 and 105 evaluations when formulation A1, A2, A3 are used, respectively.

IV.B. Short column

This problem involves the plastic analysis of a short column with rectangular cross section (width $b$ and depth $h$) having uncertain material properties (yield stress $Y$) and subject to uncertain loads (bending moment $M$ and axial force $P$). The limit state function is defined as

$$g(x) = 1 - \frac{4M}{bh^2Y} - \frac{P^2}{b^2h^2Y^2}, \quad (15)$$

The distributions for $P$, $M$ and $Y$ are $N(500,100)$, $N(2000,400)$, $N(500,100)$, and Lognormal with $(\mu, \sigma)=(5,0.5)$ respectively, between a correlation coefficient of 0.5 between $P$ and $M$. The epistemic variables are the beam width $b$ and the depth $h$ with intervals of $[5,15]$ and $[15,25]$. The area is a function of the epistemic variables, while the reliability index is a function of both aleatory and epistemetic uncertainties.

By applying the Simplex\textsuperscript{2} strategy and PC-GA, the obtained converged intervals $\beta_{CDF}$ are equal to $[-2.2612,11.9552]$ and $[-2.2814,12.0437]$, respectively. These values are coherent with reference solutions\textsuperscript{14} in literature. A convergence rate is then calculated by computing the error with respect to the reference solution as a function of the number of simulations. In figure 3, we report the convergence rates for each strategy in terms of $L^\infty$ metrics on $\beta_{CDF}$ intervals. Simplex\textsuperscript{2} displays a very higher convergence rate than PC-GA, i.e. a reduction of number of sample of three order of magnitude is observed in order to reach the same order of error. The formulation A1, i.e. the rude coupling of SSC and NM, allows obtaining a very strong reduction of the computational cost. Probably, this is related to the adaptive strategy on which SSC is built with respect to the classical PC. Between the three formulations, A3 permits a reduction of nearly 30% of the computational cost with respect to A1 at a fixed error. Remark also that all the strategies converge more rapidly when minimizing $\beta$.

IV.C. Cantilever beam

This problem involves the uniform cantilever beam reported in Ref.\textsuperscript{14}. Four uncertainties with normal distribution are taken into account, i.e. the yield stress $R$ and the Youngs modulus $E$ of the beam material and the horizontal and vertical load $X$ and $Y$, using $N(40000,2000)$, $N(2.9E+7,1.45E+6)$, $N(500,100)$, $N(1000,100)$ respectively. The constants $L$ and $D$ are equal to 100 in and 2.2535 in, respectively. The stress $S$ and the displacement $D$ assume the following form:
Figure 3. Convergence rates for the Simplex$^2$ and the PC-GA in the short column test function.

\begin{align}
S &= \frac{600}{w^2 t} Y + \frac{600}{w^2 t} X \leq R, \quad (16) \\
D &= \frac{4L^3}{E wt} \left( \frac{Y}{t^2} \right)^2 + \left( \frac{X}{w^2} \right)^2 \leq D_0. \quad (17)
\end{align}

If we indicate with \( g_s = S - R \) and \( g_D = D - D_0 \), negative \( g \) values represent safe regions of the parameters space.

The epistemic variables are the beam width \( w \) and the thickness \( t \) with intervals of \([1,10]\). The area \( wt \) is a function of the epistemic variables, while the reliability indices are functions of both aleatory and epistemic variables. The Simplex\(^2\) converge with a \( \beta_{CCDF_s} \) interval of \([-10.3801, 19.8561]\) and a \( \beta_{CCDF_D} \) interval of \([-9.6405, 1356.58]\). Convergence rates in \( L^\infty \) norm for \( \beta_{CCDF_s} \) are reported in figure 4. Also in this test-case, Simplex\(^2\) method allows obtaining converged solutions with a strong reduction of the computational cost with respect to PC-GA. Remark that in this case, slighter differences are observed among the formulations than in the previous case, though A3 displays always the best performances. Here, the use of A3 allows a reduction of the computational cost of 5% with respect to the decoupled formulation A1.

V. Conclusion

In this paper, an efficient numerical method based on Simplex representation is proposed for handling mixed aleatory-epistemic uncertainty. An interval analysis approach is taken into account, permitting to convert the stochastic problem in a robust optimization problem. The use of the Simplex\(^2\) algorithm, i.e. the strong coupling between the stochastic space representation and the Nelder-Mead algorithm, allows reducing the global number of simulations required to attain convergence. This method is tested on several algebraic benchmark test-cases displaying good properties in terms of accuracy and computational cost. This method appears very attractive for epistemic uncertainty analysis, though more work is demanded to improve the methodology.

References

Figure 4. Convergence rates for the Simplex and the PC-GA in the cantilever test function.


