SIMPLEX-SIMPLEX APPROACH FOR ROBUST DESIGN OPTIMIZATION

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Abstract. The Simplex Stochastic Collocation (SSC) method has been developed for adaptive uncertainty quantification (UQ) in computational problems with random inputs. In this work, we show how this formulation based on Simplex space representation, discretization of non-hypercube probability spaces and adaptive refinements can be easily coupled with a well-known optimization method, i.e. Nelder-Mead algorithm, also known as Downhill Simplex Method. Numerical results show that this method is very efficient for mono-objective optimization and minimizes global number of deterministic evaluations in order to determine optimal design. This method is then applied to a realistic problem of robust optimization of a two-component race-car airfoil.

Key words: robust optimization, uncertainty quantification, simplex method, Nelder-Mead.

1 INTRODUCTION

In most engineering applications, the use of deterministic models during optimization process is no more considered a good strategy in order to obtain reliable designs. A design procedure must explicitly take into account all the system uncertainties, because performance of designs can be significantly change. The risk is to obtain final designs with good performances at the design point but poor off-design characteristics, that is well-known in literature as over-optimization. Aim of robust design optimization is to determine a design which is relatively insensitive with respect to physical and modeling uncertainties.

Robust optimization processes may require a prohibitive computational cost when dealing with a large number of uncertainties and a highly non-linear fitness function. Efforts in the development of numerical method are directed mainly to reduce the number of deterministic evaluations necessary for optimization problem and for the stochastic performance evaluation1. The overall cost is typically the product of the cost of the two approaches because the stochastic analysis and the optimization strategy are completely decoupled. Decoupled approaches are simple but more expensive than necessary. In this paper, we propose a multi-scale strategy based on simplex space representation in order to minimize global cost of the robust design. This reduction is obtained i) by a coupled stopping criterion and ii) by adaptive element refinements with polynomial extrapolation
to the boundaries of the probability and design domain. The paper is organized as follows. Section 2 is devoted to the description of the numerical method proposed for performing robust optimization. Section 3 describes analysis and results obtained on several test-cases and on a robust optimization study of a multi-component airfoil.

2 METHODOLOGY

Consider the following computational problem for an output of interest \( u(x, y, t, \xi(\omega)) \)

\[
\mathcal{L}(x, y, t, \xi(\omega); u(x, y, t, \xi(\omega))) = S(x, y, t, \xi(\omega)),
\]

with appropriate initial and boundary conditions.

The operator \( \mathcal{L} \) and source term \( S \) are defined on domain \( D \times T \times \Xi \), where \( x \in D \) and \( t \in T \) are the spatial and temporal coordinates with \( D \subset \mathbb{R}^d \), \( d \in \{1, 2, 3\} \), and \( T \subset \mathbb{R} \). The vector \( y \) is defined on a domain \( D' \) and represents the vector of design variables, where \( N \) is the number of design variables. Randomness is introduced in (1) and its initial and boundary conditions in terms of \( n_\xi \) second-order random parameters \( \xi(\omega) = \{\xi_1(\omega_1), \ldots, \xi_{n_\xi}(\omega_{n_\xi})\} \in \Xi \) with parameter space \( \Xi \subset \mathbb{R}^{n_\xi} \). The symbol \( \omega = \{\omega_1, \ldots, \omega_{n_\xi}\} \in \Omega \subset \mathbb{R}^{n_\xi} \) denotes events in the complete probability space \( (\Omega, \mathcal{F}, P) \) with \( \mathcal{F} \subset 2^\Omega \) the \( \sigma \)-algebra of subsets of \( \Omega \) and \( P \) a probability measure. The random variables \( \omega \) are by definition standard uniformly distributed as \( U(0, 1) \). Random parameters \( \xi(\omega) \) can have any arbitrary probability density \( f_\xi(\xi(\omega)) \). The argument \( \omega \) is dropped from here on to simplify the notation. The objective of uncertainty propagation is to find the probability distribution of \( u(x, y, t, \xi) \) and its statistical moments \( \mu_{ui}(x, y, t) \) given by

\[
\mu_{ui}(x, y, t) = \int_{\Xi} u(x, y, t, \xi)^i f_\xi(\xi) d\xi,
\]

where statistical moments are then dependent on the vector of design variables \( y \), i.e. \( \mu_{ui}(x, y, t) \). A minimization problem can be formulated as

\[
\min_{y \subset D'} \mu_{ui}(x, y, t),
\]

where the process of finding solutions of equations (1) and (3) is referred to as robust design optimization.

A local UQ method computes these weighted integrals over parameter space \( \Xi \) as a summation of integrals over \( n_\xi \) disjoint subdomains \( \Xi = \bigcup_{j=1}^{n_\xi} \Xi_j \)

\[
\mu_{ui}(x, y, t) = \sum_{j=1}^{n_\xi} \int_{\Xi_j} u(x, y, t, \xi)^i f_\xi(\xi) d\xi.
\]

In the SSC approach\(^2\) the integrals in the simplex elements \( \Xi_j \) are computed by approximating response surface \( u(\xi) \) by an interpolation \( w(\xi) \) of \( n_\xi \) samples \( v = \{v_1, \ldots, v_{n_\xi}\} \).

Here the arguments \( x, y \) and \( t \) are omitted to simplify the notation.

The non-intrusive SSC\(^2\) uncertainty quantification method \( q \) then consists of a sampling method \( g \) and an interpolation method \( h \), for which holds \( w(\xi) = q(u(\xi)) = h(g(u(\xi))) \).
The sampling method \( g \) selects the sampling points \( \xi_k \) for \( k = 1, \ldots, n_s \) and returns the sampled values \( v = g(u(\xi)) \), with \( v_k = g_k(u(\xi)) = u(\xi_k) \).

Sample \( v_k \) is computed by solving (1) for realization \( \xi_k \) of the random parameter vector \( \xi \)

\[
L(x, y, t; \xi_k; v_k(x, y, t)) = S(x, y, t, \xi_k), \quad (5)
\]

for \( k = 1, \ldots, n_s \).

The interpolation of the samples \( w(\xi) = h(v) \) consists of a piecewise polynomial function

\[
w(\xi) = w_j(\xi), \quad \text{for } \xi \in \Xi_j, \quad (6)
\]

with \( w_j(\xi) \) a polynomial interpolation of degree \( p \) of the samples \( v_j = \{v_{k_j,0}, \ldots, v_{k_j,N}\} \) at the sampling points \( \{\xi_{k_j,0}, \ldots, \xi_{k_j,N}\} \) in element \( \Xi_j \), where \( k_{j,l} \in \{1, \ldots, n_s\} \) for \( j = 1, \ldots, n_e \) and \( l = 0, \ldots, N \), with \( N \) the number of samples in the simplexes.

The polynomial interpolation \( w_j(\xi) \) in element \( \Xi_j \) can then be expressed in terms of a truncated Polynomial Chaos expansion

\[
w_j(\xi) = \sum_{m=0}^{P} c_{j,m} \Psi_{j,m}(\xi). \quad (7)
\]

where the polynomial coefficients \( c_{j,m} \) can be determined from the interpolation condition

\[
w_j(\xi_{k_{j,l}}) = v_{k_{j,l}}, \quad (8)
\]

for \( l = 0, \ldots, N \), which leads to a matrix equation, that can be solved in a least-squares sense for \( N > P \).

The probability distribution function and the statistical moments \( \mu_{u_i} \) of \( u(\xi) \) given by (4) are then approximated by the probability distribution and the moments \( \mu_{w_i} \) of \( w(\xi) \)

\[
\mu_{u_i}(x, y, t) \approx \mu_{w_i}(x, y, t) = \int_{\Xi} w_j(x, y, t, \xi) f_\xi(\xi) d\xi, \quad (9)
\]

in which the multi-dimensional integrals are evaluated using a weighted Monte Carlo integration of the response surface approximation \( w(\xi) \) with \( n_{mc} \gg n_s \) integration points.

This is a fast operation, since it only involves integration of piecewise polynomial function \( w(\xi) \) given by (6) and does not require additional evaluations of the exact response \( u(\xi) \).

In order to solve (3), we used the Nelder-Mead (NM) Method\(^3\). This method uses also a simplex space representation, constituted by \( N + 1 \) vertices if \( N \) is the dimension of vector \( y \). It generates new designs by extrapolating the behavior of the objective function measured at each one of the basic designs, that constitute the geometric simplex. The algorithm then chooses to replace one of these design with the new one and so on.
For all details concerning implementation of the basic NM method, we refer to classical reference 4.

The present Simplex-Simplex (hereafter referred to as Simplex$^2$) method is an efficient multi-scale coupling of SSC and NM method, based on two different levels of the simplex. For each design variable, a stochastic simplex (micro-scale) is generated using SSC method. The set of design variables constitutes the so-called geometric simplex (macro-scale). In the formulation of the algorithms and the examples we show below we assume that the uncertain variables and the design parameters are independent for simplicity. It is possible to use the same approach for cases in which some of the design variables are uncertain; this will be explored in future work.

This method has several advantages. First one, a stopping criterion based on the minimal error in both the stochastic and geometric simplex can be defined. Second, it is possible to exploit interpolating polynomials $w_j(\xi)$ (6) both for stochastic simplex and geometric simplex. Then Nelder-Mead Method could be accelerated with respect to the classical version by using this response surface. In the following section, a detailed description of the algorithm is provided.

3 SIMPLEX$^2$ ALGORITHM

The Simplex$^2$ algorithm can be summarized as follows. The optimization space is named as $\Xi_{opt}$, and an element $j$ of $\Xi_{opt}$ as $\Xi_{opt,j}$. For each design point $y_o$, we define a probabilistic space $\Xi_o$. The notation $\Xi_{o,j}$ refers to an element $j$ of stochastic simplex associated to $y_o$. Let us suppose to maximize (minimize) a given function constituted by statistics of a given output, for example to maximize mean $\mu$.

1. The initial grid of sampling designs $y_o$ is composed of the $2^n y$ vertexes of the hypercube enclosing the optimization space $\Xi_{opt}$ and one sampling point in the interior. For each sampling design $y_o$, the following steps are followed.

   (a) An initial grid of sampling points $\xi_k$ is composed of the $2^n \xi$ vertexes of the hypercube enclosing the probability space $\Xi_o$ and one sampling point in the interior.

   (b) The $n_{s\text{\scriptsize init}}$ initial samples $v_k$ are computed by solving $n_{s\text{\scriptsize init}}$ deterministic problems (1) for the parameter values corresponding to the initial sampling points $\xi_k$ located in $\Xi_o$ only.

   (c) The initial discretization of parameter space $\Xi_o$ is constructed by making a Delaunay triangulation of all sampling points $\xi_k$ resulting in $n_e$ simplex elements $\Xi_j$.

   (d) The polynomial approximation $w_j(\xi)$ in each of the interpolation elements $\Xi_j$ is constructed.

   (e) The statistical moments $\mu_{u_i}$ of $w(\xi)$, and the error computed on this quantity (see Ref. 2 for more details) are calculated by Monte Carlo integration of (4). If the error is over a given value, then a refinement on the stochastic simplex is performed. The criterion for the refinement is the following

   \[
   \text{error}_{SSC} = \mu_{u_i}(y_N) - \mu_{u_i}(y_1)
   \]

2. For each sampling design $y_o$, $\mu_{u_i}$ is known.
3. The polynomial interpolation $w_j(y)$ (7) on $\Xi_{opt}$ is then constructed (denoted P1 in the following), by solving (7) on $\Xi_{opt}$.

4. Order the fitness function according to the values of $\mu_i$.

$$\mu_{i_1}(y_1) < \mu_{i_2}(y_2) < \ldots < \mu_{i_n}(y_n)$$

(11)

5. Compute the center of gravity $y_0$ of all points expect $y_{n+1}$.

6. Compute reflected point by means of the response surface P1, i.e. $y_r = y_0 + \alpha(y_0 - y_{n+1})$ with $\alpha = 1$. If the reflected point is better than the second worst, but not better than the best, i.e. $\mu_{i_1}(y_1) < \mu_{i_2}(y_r) < \ldots < \mu_{i_n}(y_n)$, then obtain a new simplex by replacing the worst point $y_{n+1}$ with the reflected point $y_r$, and go from 1-a to 1-e in order to compute $\mu_{i_n}(y_r)$.

7. If the reflected point is the point so far, $\mu_{i_1}(y_r) < \mu_{i_2}(y_1)$, then compute by means of the response surface P1, i.e. the expanded point $y_e = y_0 + \Gamma(y_0 - y_{n+1})$ with $\Gamma = 2$. If the expanded point is better than the reflected point, $\mu_{i_1}(y_e) < \mu_{i_2}(y_r)$, then obtain a new simplex by replacing the worst point $y_{n+1}$ with the expanded point $y_e$, and go from 1-a to 1-e compute exactly $\mu_{i_n}(y_e)$. Else obtain a new simplex by replacing the worst point $y_{n+1}$ with the reflected point $y_r$, and go to step 6. Else (i.e. reflected point is not better than second worst) continue at step 9.

8. Here, it is certain that $\mu_{i_1}(y_r) > \mu_{i_n}(y_n)$. Compute contracted point by means of the response surface P1, i.e. $y_c = y_{n+1} + \rho(y_0 - y_{n+1})$ with $\rho = 0.5$. If the contracted point is better than the worst point, i.e. $\mu_{i_1}(y_c) < \mu_{i_2}(y_{n+1})$ then obtain a new simplex by replacing the worst point $x_{n+1}$ with the contracted point $y_c$, and go from 1-a to 1-e compute exactly $\mu_{i_1}(y_c)$. Else go to step 9.

9. For all but the best point, replace the point with $y_i = y_1 + \sigma(y_i - y_1)$ with $\sigma = 0.5$. Go from 1-a to 1-e to compute exactly $\mu_{i_1}(y_i)$.

4 RESULTS

4.1 Rosenbrock stochastic problem

There exist several stochastic formulations of Rosenbrock optimization problem. In this work, we propose a slightly modified version, where a non-linear dependence on stochastic variable is taken into account. More precisely, we consider the following function

$$f(x) = \sum_{i=1}^{N-1} \left[ (1 - x_i)^2 + 100\sqrt{\varepsilon_i + \alpha} (x_{i+1} - x_i^2)^2 \right]$$

(12)

where $\varepsilon_i$ varies in a uniform random variable in $[0, 1]$ and $\alpha$ is taken equal to 1. This stochastic function has the same global optimum of the its deterministic counterpart at $(1,1,1,\ldots)$.

We show results obtained by comparing three different formulations. In the first one, called A1, SSC and NM methods have been used in a decoupled way, i.e. NM method is used in its traditional version and SSC method is seen as a black-box. In the second formulation, called hereafter A2, the estimated error on the geometric simplex is used as
stopping criterion for the error on stochastic simplex, as shown in (10). Then, for design not close to optimum, the associated stochastic simplex will be less refined. In the third formulation (A3), the algorithm described in section 3 is used. The problem defined by (12) is considered with $N = 2$, then with two design variables and one uncertainty. First, we apply Simplex-Simplex method for minimizing $\mu(f)$. In table 1, results obtained in terms of deterministic evaluations, where $N_{it}$ is the number of iterations for NM algorithm and $N_0$ is the global number of deterministic evaluations, are reported for each formulation. Note that a DOE constituted by 25 samples in the geometric simplex is used for each formulation, then with $N_{it} = 25$ and $N_0 = 256$, and then this computational cost is not reported in table 1. The exact same optimal design is obtained (the optimal fitness function is nearly $10^{-6}$ where the optimal theoretical fitness is equal to zero) in both cases, without variations in $N_{it}$, but with a reduction of 40.8% for $N_0$ in the case A2. The same $N_{it}$ is obtained because convergence for $\mu(f)$ is more fast that convergence of the geometric simplex. The use of formulation A3 displays an important result in terms of computational cost, i.e. allowing a reduction of 66.3% for $N_0$ with respect to the decoupled formulation A1. In this case, 31 individuals are generated during the transitory steps in the Nelder-Mead algorithm (from step 6 to step 8) by using the response surface instead of a direct computation, and this explains the strong saving achieved. Simplex evolution in the design variables space is reported in figure 1. For the optimal individual, the stochastic response surface with respect to the uncertainty is nearly coincident with the exact solution, as shown in figure 2. Then the same optimization is performed in order to minimize $\mu(f) + \sigma(f)$ (DOE constituted by 25 samples in the geometric simplex is used for each formulation, then with $N_{it} = 25$ and $N_0 = 274$), and the results in terms of $N_{it}$ and $N_0$ are reported in table 2. If A2 formulation is used, $N_0$ is reduced of 42.3%, with the same $N_{it}$. Results obtained by using A3 are impressive displaying a reduction of 66.3% for $N_0$. In fact, in this case, 29 individuals are estimated by using the response surface instead of direct computations. These first results show the potentiality of the proposed approach on a classical problem of robust optimization. In the next section, a more complex algebraic functions is taken into account, i.e. the function proposed by Yang.

<table>
<thead>
<tr>
<th>Formulation</th>
<th>$N_{it}$</th>
<th>$N_0$</th>
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<tbody>
<tr>
<td>A1</td>
<td>62</td>
<td>368</td>
</tr>
<tr>
<td>A2</td>
<td>62</td>
<td>218</td>
</tr>
<tr>
<td>A3</td>
<td>31</td>
<td>117</td>
</tr>
</tbody>
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Table 1: Rosenbrock problem. Minimization of $\mu$ where $N_{it}$ is the number of iterations for NM algorithm, $N_0$ global number of deterministic evaluations.

<table>
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<th>Formulation</th>
<th>$N_{it}$</th>
<th>$N_0$</th>
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<tbody>
<tr>
<td>A1</td>
<td>62</td>
<td>383</td>
</tr>
<tr>
<td>A2</td>
<td>62</td>
<td>221</td>
</tr>
<tr>
<td>A3</td>
<td>33</td>
<td>129</td>
</tr>
</tbody>
</table>

Table 2: Rosenbrock problem. Minimization of $\mu + \sigma$ where $N_{it}$ is the number of iterations for NM algorithm, $N_0$ global number of deterministic evaluations.
4.2 Yang Problem

Yang and Deb proposed the following function for a robust optimization problem

\[ f(x) = \left[ e^{-\sum_{i=1}^{n}(x_i/\beta)^{2m}} - 2e^{-\sum_{i=1}^{n} \epsilon_i(x_i-x)^2} \right] \cdot \prod_{i=1}^{n} \cos^2 x_i, \quad m = 5. \quad (13) \]

This function has many local minima and the unique global minimum \( f_x = -1 \) at \( x_* = (\pi, \pi, \ldots) \) for \( \beta = 15 \) within the domain \(-7 \leq x_i \leq 13 \) for \( i = 1, 2, \ldots, n \). Here the random variables are uniformly distributed in (0,1). If all \( \epsilon_i \) are small (0.05), four local minima exist as shown in the iso-contour curve reported in figure 3a, while for higher values such as 0.5 only there is only one minima and the landscape is different (3b).

Formulations A1, A2 and A3 are checked on this function by minimizing \( \mu \) and \( \mu + \sigma \). A DOE constituted by 36 samples in the geometric simplex is used for each formulation, then with \( N_{it} = 36 \) and \( N_0 = 1188 \) (this computational cost is not reported in the following tables (3 and 4). The same unique global minimum is obtained in the three formulations. Table 3 shows the results obtained for \( \mu \) in terms of \( N_{it} \) and \( N_0 \). With the formulation A2 a saving of 36.5% for \( N_0 \) is obtained with respect to the decoupled formulation A1, with the same \( N_{it} \) (convergence for \( \mu(f) \) is more fast that convergence of the geometric simplex). Formulation A3 displays a reduction of 69.1% for \( N_0 \) (25 individuals are generated during the transitory steps in the Nelder-Mead algorithm (from step 6 to step 8) and evaluated by means of response surfaces). In table 4, results obtained for minimizing \( \mu + \sigma \) are reported. In this case, reductions for \( N_0 \) of 33.4% (A2) and of 60.7% (A3) with respect to A1 are obtained.
4.3 REALISTIC CASE

The proposed method is then applied to a realistic problem of robust optimization of a multi-component airfoil, where the geometry and the associated mesh are illustrated in figure 4. One design variable, $D_X$ (that varies from 0.01 to 0.08), and one uncertainty, $Z_{inlet}$ (Unif(0, 0.3)), are considered. A preliminary plan of experience on $D_X$ and $Z_{inlet}$ is performed in order to estimate the hardness of the present problem. In figures 5 and 6, lift ($C_L$) and lift-to-drag ratio ($C_L/C_D$) are reported, respectively. Magnitude of lift displays a quasi-linear dependence on $D_X$, with higher values for lower $D_X$ (figure 5). For the lift-to-drag ratio, a non-linear behavior can be observed in figure 6, where high values are obtained for low $D_X$ but also for some values close to the maximal limit.

The three formulations based on Simplex representation have been compared in terms of computational cost in order to maximize $\mu$ and $\mu + \sigma$ of $C_L$ and $C_L/C_D$. For each case, the same optimum is obtained by using A1, A2 and A3. For $C_L$, the maximal value for $\mu(C_L)$ is 0.0041343 obtained at $x = 0.016563$, while the maximal value for $\mu(C_L) - \sigma(C_L)$ is 0.0040705 that is obtained at $x = 0.020391$. For $C_L/C_D$, the maximal value for $\mu(C_L/C_D)$ ($\mu(C_L/C_D) - \sigma(C_L/C_D)$) is 28.4129 (25.5571) at $x = 0.041263$ ($x = 0.041259$).
Figure 3: Iso-contour curve for the function proposed by Yang, considering $\epsilon = 0.05$ (a) and $\epsilon = 0.5$ (b)

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<th>Formulation</th>
<th>$N_{it}$</th>
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<td>53</td>
<td>815</td>
</tr>
<tr>
<td>A2</td>
<td>53</td>
<td>543</td>
</tr>
<tr>
<td>A3</td>
<td>37</td>
<td>320</td>
</tr>
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</table>

Table 4: Yang problem. Minimization of $\mu + \sigma$ where $N_{it}$ is the number of iterations for NM algorithm, $N_0$ global number of deterministic evaluations

Results in terms of computational cost are summarized in tables 5 and 6 for $C_L$ and in tables 7 and 8 for $C_L/C_D$. With respect to the decoupled formulation A1, using A3 (A2) permits a reduction of 42.7% (29.3%) for $\mu(C_L)$ and of 41.0% (27.8%) for $\mu(C_L) - \sigma(C_L)$. Concerning $C_L/C_D$, a reduction of 53.2% (31.8%) for $\mu(C_L/C_D)$ and of 50.5% (31.2%) for $\mu(C_L/C_D) - \sigma(C_L/C_D)$ have been obtained by using A3 (A2) with respect to A1. These trends display the good performances of the proposed method also for a realistic case.

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<tbody>
<tr>
<td>A1</td>
<td>13</td>
<td>550</td>
</tr>
<tr>
<td>A2</td>
<td>13</td>
<td>389</td>
</tr>
<tr>
<td>A3</td>
<td>10</td>
<td>315</td>
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Table 5: CFD problem. Maximization of $\mu(C_L)$ where $N_{it}$ is the number of iterations for NM algorithm, $N_0$ global number of deterministic evaluations
Table 6: CFD problem. Maximization of $\mu(C_L) - \sigma(C_L)$ where $N_{it}$ is the number of iterations for NM algorithm, $N_0$ global number of deterministic evaluations

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<th>Formulation</th>
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<th>$N_0$ (5-248)</th>
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<tr>
<td>A1</td>
<td>15</td>
<td>748</td>
</tr>
<tr>
<td>A2</td>
<td>15</td>
<td>540</td>
</tr>
<tr>
<td>A3</td>
<td>12</td>
<td>442</td>
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5 CONCLUSIONS

In this paper, we proposed a multi-scale strategy based on simplex space representation in order to minimize global cost of the robust design. In particular, we coupled the simplex stochastic colocation method for uncertainty quantification with the well-known Nelder-Mead optimization algorithm. The efficiency of the former is based on high degree polynomial interpolation, randomized refinement sampling, and Essentially Extremum Diminishing (EED) extrapolation in a Delaunay triangulation of parameter space, while the basic properties of Nelder-Mead algorithm have been improved accelerating some evaluations by means of response surface based on the high degree polynomial interpolation of the geometric simplex. A strong reduction of the computational cost is obtained i) by a coupled stopping criterion and ii) by adaptive element refinements with polynomial extrapolation to the boundaries of the probability and design domain. This method has been applied to two algebraic benchmark test cases, Rosenbrock problem and a complex function proposed by Yang. By using the same initial plan of experience, a re-
Figure 5: Lift contours for the multi-component airfoil in the plan $D_X - Z_{inlet}$

Table 7: CFD problem. Maximization of $\mu(C_L/C_D)$ where $N_{it}$ is the number of iterations for NM algorithm, $N_0$ global number of deterministic evaluations

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<tr>
<td>A1</td>
<td>24</td>
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<td>A2</td>
<td>24</td>
<td>819</td>
</tr>
<tr>
<td>A3</td>
<td>15</td>
<td>562</td>
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</table>

duction for the global number of deterministic evaluations of 66.3% both for $\mu$ and $\mu + \sigma$ is obtained for Rosenbrock, and of 69.1% (for $\mu$) and 60.7% (for $\mu + \sigma$) is obtained for the function proposed by Yang, with respect to the decoupled formulation A1. Finally, the proposed approach has been applied to a realistic problem of robust optimization of a two-component race-car airfoil in order to maximize lift and lift-to-drag ratio. Also in this case, a reduction of 42.7% for $\mu(C_L)$ and of 41.0% for $\mu(C_L) - \sigma(C_L)$, and a reduction of 53.2% for $\mu(C_L/C_D)$ and of 50.5% for $\mu(C_L/C_D) - \sigma(C_L/C_D)$ have been obtained.

REFERENCES


Figure 6: Lift-to-drag ratio contours for the multi-component airfoil in the plan $D_X - Z_{inlet}$

Table 8: CFD problem. Maximization of $\mu(C_L/C_D) + \sigma(C_L/C_D)$ where $N_{it}$ is the number of iterations for NM algorithm, $N_0$ global number of deterministic evaluations

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<tr>
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<td>22</td>
<td>757</td>
</tr>
<tr>
<td>A3</td>
<td>16</td>
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</tr>
</tbody>
</table>
