Comparison of Stochastic Collocation Methods for Uncertainty Quantification of the Transonic RAE 2822 Airfoil

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Abstract:
Non–intrusive Simplex Elements Stochastic Collocation, Stochastic Collocation with Clenshaw–Curtis points, and the Padé–Legendre method are applied to the transonic RAE 2822 airfoil test case of the NODESIM Workshop on Quantification of CFD Uncertainties. Uncertainty in a combination of the free stream Mach number $M_\infty$, angle of attack $\alpha$, and thickness–to–chord ratio $t/c$ described by uniform and normal probability distributions is propagated to the mean, standard deviation, and probability density functions of the lift $C_l$, drag $C_d$, and pitching moment $C_m$ coefficients. In addition the uncertainty bars on the mean surface pressure coefficient $C_p$, and the mean and standard deviation fields for the static pressure $p$ are presented.

The analysis of the relative importance of the random input parameters shows that the effect of none of the parameters is negligible compared to the other parameters for the considered outputs of interest.

1. Introduction

The European Sixth Framework Programme research project NODESIM–CFD on Non–Deterministic Simulation for CFD–Based Design Methodologies [7] successfully coordinated development and application of uncertainty quantification (UQ) methods over the last years. At its concluding Workshop on Quantification of CFD Uncertainties [8] results for selected test problems were compared as submitted by contributors throughout Europe and beyond. The mandatory test case for the external flow category was the transonic flow over the RAE 2822 airfoil subject to three uncertainties in the Mach number, angle of attack, and thickness–to–chord ratio with a uniform and normal distribution. Transonic flow simulations are of particular interest for UQ because of the usually sensitive shock wave location which can lead to amplification of input variability to outputs of interest. The test problem poses specific difficulties for UQ methods due to the discontinuity in the pressure field in combination with smooth response surfaces for integral quantities of lift, drag, and pitching moment. The compared UQ methods are therefore assessed on their ability to both approximate smooth response surfaces in multi–dimensional probability spaces efficiently and at the same time maintain robustness in the presence of discontinuities.

The increased attention for UQ methodologies originates from the experience that currently available methods are inadequate for application to computational fluid dynamics (CFD) problems. For example the classical Monte Carlo technique, which is based on performing a large number of random
computations, is impractical for CFD problems which can already be computationally intensive in the deterministic sense. On the other hand, the more advanced Stochastic Collocation method [2] based on Gauss quadrature sampling and Lagrangian polynomial interpolation in parameter space has been shown to have difficulty approximating higher-dimensional probability spaces and discontinuous responses. Sparse grid approaches [12, 22] have been considered as efficient alternatives for the standard multi-dimensional formulation of the Stochastic Collocation method based on tensor products. Also separated solution approximations have been developed to achieve a linear increase of computational costs with dimension [5].

For the more robust approximation of discontinuous responses, multi-element Stochastic Collocation [6] and Stochastic Galerkin [13] methods have been proposed. These approaches are usually based on employing a single element method independently in multiple hypercubes discretizing the probability space. For higher order interpolations these methods can still result in local oscillations and overshoots can be present even for low approximation orders. Often not all samples in an element can be reused after refinement and tensor product extensions to higher dimensions are required which compromises the efficiency of multi-element discretizations.

Motivated by the RAE 2822 test case, we develop in this paper a Simplex Elements Stochastic Collocation (SESC) method which combines a robust approximation of discontinuous responses with an efficient discretization in multi-dimensional probability spaces. The SESC method is an extension of the simplex elements method with Newton–Cotes quadrature [20] to higher order interpolation and randomized sampling. Results are compared with Stochastic Collocation (SC) based on Clenshaw–Curtis quadrature and the Padé–Legendre (PL) method [3]. The Padé approximation of the response by a global rational function through the collocation points gives potentially a more reliable approximation of highly non-linear response surfaces than SC. The geometric uncertainty in the thickness-to-chord ratio is handled using a general purpose explicit mesh deformation method based on Inverse Distance Weighting (IDW) interpolation [21] of the surface displacements to the spatial grid.

The presentation of the test case results is organized as follows. After the definition of the RAE 2822 problem in section 2, the deterministic flow simulation is analyzed in standard mesh convergence, verification, and validation studies in section 3. In section 4 the applied non-intrusive uncertainty quantification methods are described in separate paragraphs for Simplex Elements Stochastic Collocation, Stochastic Collocation, the Padé–Legendre method, and the Inverse Distance Weighting mesh deformation approach. The uncertainty quantification results of the three methods are presented and compared in section 5. A discussion of the main findings concludes the paper in section 6.

2. RAE 2822 airfoil test case

The geometry of the RAE 2822 airfoil shown in Figure 1 is described by the design airfoil coordinates tabulated in [4] with a maximal thickness-to-chord ratio of \( t/c = 0.1211 \). The airfoil is designed using a second order method [1] for a free stream Mach number of \( M_\infty = 0.66 \) and lift coefficient \( C_l = 0.56 \) at angle of attack \( \alpha = 1.06^o \). The off–design nominal flow conditions considered here correspond to free stream Mach number \( M_\infty = 0.734 \), angle of attack \( \alpha = 2.79^o \), and Reynolds number \( Re = 6.5 \cdot 10^6 \).

Uncertainties are imposed on the Mach number \( M_\infty \), angle of attack \( \alpha \), and thickness-to-chord ratio \( t/c \) with standard deviations \( \sigma_M = 0.005, \sigma_\alpha = 0.1, \) and \( \sigma_{t/c} = 0.005 \), respectively. Both uniform and normal distributions are considered for the three independent random input parameters, see Figure 2. These random inputs are selected based on expert opinions on realistic variations in practical operating conditions. The Reynolds number is kept fixed at \( Re = 6.5 \cdot 10^6 \) during the stochastic simulations, since
different implementations for varying the Mach number $M_\infty$, by for example changing the free stream velocity or the free stream speed of sound, would differently affect the Reynolds number. In order to maintain a uniquely defined problem specification we correct the dynamic viscosity $\mu$ with varying free stream velocity to change the Mach number $M_\infty$ at constant Reynolds number. The dynamic viscosity is then taken constant throughout the flow field. The angle of attack is varied by changing the inflow boundary condition.

The effect of the input variability on the pressure part of the lift $C_l$, drag $C_d$, and pitching moment $C_m$ coefficients is resolved in terms of the mean, standard deviation, and probability density function. The moment reference point is $(x, y) = (0.25, 0)$ in the airfoil coordinate system of Figure 1. In addition the mean and uncertainty bars of the surface pressure distribution $C_p$, and the mean and standard deviation of the two–dimensional pressure field $p(x, y)$ are computed. The relative importance of the uncertain parameters is also determined.

3. Deterministic simulation

A suitable computational mesh for the stochastic simulations is selected based on a mesh convergence study for the deterministic RAE 2822 problem. The deterministic results are verified and validated by comparison with available numerical and experimental data.

3.1. Mesh convergence study

The deterministic simulations are based on solving the Reynolds–Averaged Navier–Stokes (RANS) equations on a structured hexahedral C–type mesh using the in–house RANS solver Joe [15]. A second order spatial discretization is used in combination with the minmod limiter and the Spalart–Allmaras turbulence model. Dual time integration is performed by implicit Euler integration until a convergence criterion of $10^{-5}$ is reached where the linearized system is solved by the PETSC-GMRES algorithm.

The analysis of the deterministic RAE 2822 problem is performed for the flow conditions of experiment number 6 from [4] $M_\infty = 0.725$, $\alpha = 2.92^\circ$, and $Re = 6.5 \cdot 10^6$, which are closest to the mean conditions of the stochastic problem $M_\infty = 0.734$, $\alpha = 2.79^\circ$, and $Re = 6.5 \cdot 10^6$. The wind tunnel conditions are corrected for external flow computations in an earlier validation study at NASA [17] to
Figure 2: Input probability density function.
The finest mesh of 112000 volumes considered in the mesh convergence study is shown in Figure 3. A comparison with a coarser mesh of 58800 volumes with a double cell size in the direction tangential to the airfoil surface is presented here. The resulting pressure coefficient \( C_p \) along the airfoil given in Figure 4 for the coarser and finer mesh shows only minor differences. The pressure distribution shows a weak transonic shock wave on the upper surface around \( x/c = 0.54 \). The coarser mesh gives small oscillations near the leading edge suction peak, which are absent on the finer mesh. The integrated lift \( C_l \), drag \( C_d \), and pitching moment \( C_m \) coefficients of the two meshes are given in Table 1 for the considered flow conditions \( M_\infty = 0.729 \) and \( \alpha = 2.31^\circ \). The lift and drag forces are defined as the decomposition of the total aerodynamic pressure force vector perpendicular to and in the direction of the random free stream velocity direction, respectively. The values of the coefficients for the two computations are sufficiently converged for the stochastic simulations. The finer mesh is therefore selected for the uncertainty quantification study.

\[
M_\infty = 0.729, \ \alpha = 2.31^\circ, \ \text{and} \ \text{Re} = 6.5 \cdot 10^6. \ \text{The latter flow conditions are used for the deterministic computations in this section.}
\]

The finest mesh of 112000 volumes considered in the mesh convergence study is shown in Figure 3. A comparison with a coarser mesh of 58800 volumes with a double cell size in the direction tangential to the airfoil surface is presented here. The resulting pressure coefficient \( C_p \) along the airfoil given in Figure 4 for the coarser and finer mesh shows only minor differences. The pressure distribution shows a weak transonic shock wave on the upper surface around \( x/c = 0.54 \). The coarser mesh gives small oscillations near the leading edge suction peak, which are absent on the finer mesh. The integrated lift \( C_l \), drag \( C_d \), and pitching moment \( C_m \) coefficients of the two meshes are given in Table 1 for the considered flow conditions \( M_\infty = 0.729 \) and \( \alpha = 2.31^\circ \). The lift and drag forces are defined as the decomposition of the total aerodynamic pressure force vector perpendicular to and in the direction of the random free stream velocity direction, respectively. The values of the coefficients for the two computations are sufficiently converged for the stochastic simulations. The finer mesh is therefore selected for the uncertainty quantification study.

**Table 1: Force and moment coefficients of the deterministic simulations.**

<table>
<thead>
<tr>
<th>( M_\infty = 0.729, \ \alpha = 2.31^\circ )</th>
<th>( M_\infty = 0.734, \ \alpha = 2.79^\circ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_l )</td>
<td>( C_l )</td>
</tr>
<tr>
<td>coarser mesh</td>
<td>finer mesh</td>
</tr>
<tr>
<td>( 7.092 \cdot 10^{-1} )</td>
<td>( 7.103 \cdot 10^{-1} )</td>
</tr>
<tr>
<td>( C_d )</td>
<td>( C_d )</td>
</tr>
<tr>
<td>( 4.821 \cdot 10^{-2} )</td>
<td>( 4.910 \cdot 10^{-2} )</td>
</tr>
<tr>
<td>( C_m )</td>
<td>( C_m )</td>
</tr>
<tr>
<td>( -8.947 \cdot 10^{-2} )</td>
<td>( -8.907 \cdot 10^{-2} )</td>
</tr>
</tbody>
</table>
3.2. Verification

For further verification of the deterministic results the pressure distribution is compared with the NASA study of RAE 2822 case 6 [17] mentioned above. The surface pressure coefficient here computed with Joe is compared in Figure 5 with results from the WIND and NPARC codes available at [17]. The NASA computations are performed for the Spalart–Allmaras turbulence model using local time stepping to march to a steady state on a two–dimensional single–block C–grid of 369 × 65 volumes. The numerical results show overall a good agreement, where Joe corresponds more closely to the WIND results near the shock wave and matches better with the NPARC data closer to the leading edge. It is remarked that the NASA simulations are based on the measured ordinates of the airfoil in the wind tunnel campaign reported in [4] instead of the design geometry used here.

3.3. Validation

The experimental data for case 6 from [4] is used for the validation of the numerical results. The surface pressure hole measurements agree well with the computational data in Figure 6. The small differences can be caused by the difference between the design and measured geometry of the airfoil, and the correction of the flow conditions for external flow simulations. The uncertainty in the experimental data is given by ∆C_p < ±0.0026 [4], which is too small to include in Figure 6 in terms of experimental uncertainty bars. In the following, realistic variations in Mach number, angle of attack, and thickness–to–chord ratio for practical operating conditions are propagated numerically to quantify the confidence level in the computational results in terms of numerical uncertainty bars.

The mean values for the uncertainty quantification study deviate slightly from the deterministic flow conditions considered in this section. Figure 7(a) shows that the surface pressure distributions computed at the deterministic conditions M_∞ = 0.729 and α = 2.31°, and the mean stochastic values M_∞ = 0.734 and α = 2.79° are comparable. In the mean stochastic case the shock wave is slightly stronger at a more downstream location around x/c = 0.565 with a higher negative pressure upstream of the shock location.
Also the force and moment coefficients for the two flow conditions given in Table 1 show reasonable agreement, which justifies the verification and validation (V&V) efforts for case 6. The pressure field for the mean input values is given in Figure 7(b) as reference for the stochastic results.

4. Uncertainty quantification methods

In this section the non–intrusive uncertainty quantification methods used to obtain the stochastic outputs are described. Simplex Elements Stochastic Collocation, Stochastic Collocation, and the Padé–Legendre method are considered for propagating the input variations. The geometric uncertainty in the thickness–to–chord ratio is treated using Inverse Distance Weighting mesh deformation. The approaches for determining the relative importance of the random input parameters are discussed at the presentation of the results.

4.1. Simplex Elements Stochastic Collocation

The SESC method is an extension of the simplex elements discretization of probability space based on Newton–Cotes quadrature [20] to higher order interpolation and randomized sampling. The discretization starts by sampling the vertices of the hypercube probability space and one sample in the interior. The simplex elements discretization through the samples is constructed by using a Delaunay triangulation. This triangulation maximizes the minimum angles to avoid skewed simplices by imposing that no sample may lie inside the circumcircle of another simplex. The simplex elements are refined based on the probability contained by the elements which is its volume in probability space. The extension to more advanced refinement criteria needs further attention. At a refinement a sample is added randomly in the element with highest probability. In order to obtain a good spread of the samples the new sampling point is restricted to a sub–simplex of which its vertices are defined by the middle of the \( n – 1 \) faces of the \( n \)–simplex. A level two sub–simplex used here for the sampling leads to the example of the discretization.
A piecewise linear interpolation by linear interpolation in the elements then satisfies the total variation diminishing (TVD) robustness criterion \[19\] in probability space. Higher order interpolation is obtained by using the samples at the vertices of neighboring elements based on a nearest neighbor search to construct the local response surface approximation. The local polynomials $P_i(\xi)$ in for example a one-dimensional probability space can be written as

$$P_i(\xi) = \sum_{j=0}^{p} c_{i,j} \Psi_j(\xi), \quad (1)$$

where $\Psi_j(\xi)$ are the basis polynomials of order $j$. The interpolation is made extremum diminishing (ED) by decreasing the polynomial order locally if the interpolation as an extremum in the interior of the element. This is always possible since the piecewise linear interpolation satisfies the ED property by definition. It leads to a low order robust approximation of discontinuities and a high order interpolation for smooth response surfaces. The extremum diminishing property extended to probability space is defined as \[19\]

$$\min_{\Xi}(w(\xi)) \geq \min_{\Xi}(u(\xi)) \land \max_{\Xi}(w(\xi)) \leq \max_{\Xi}(u(\xi)), \quad (3)$$

with $v_{i,j} = u(\xi_{i,j})$ the stencil of deterministic samples for element $\Xi_i$ and $\xi_{i,j}$ the corresponding sampling points. The interpolation is made extremum diminishing (ED) by decreasing the polynomial order locally if the interpolation as an extremum in the interior of the element. This is always possible since the piecewise linear interpolation satisfies the ED property by definition. It leads to a low order robust approximation of discontinuities and a high order interpolation for smooth response surfaces.
Figure 7: Deterministic results for $M_\infty = 0.734$ and $\alpha = 2.79^\circ$. 

(a) Comparison with $M_\infty = 0.729$ and $\alpha = 2.31^\circ$ 

(b) Pressure field
with exact response surface $u(\xi)$ and approximation $w(\xi)$ in the space $\Xi$ of random parameters $\xi$. The ED concept guarantees that the method generates no unphysical predictions due to overshoots at discontinuities in probability space. In multiple dimensions this robustness property is of particular importance, since it is not trivial to inspect the reliability of a multi-dimensional response surface approximation visually.

This formulation results in high flexibility where an arbitrary number of samples can be used by adding one sample at the time. The polynomial order of the interpolation is also independent of the number of samples for a sufficiently large sample size. For higher dimensional probability spaces the average number of samples per element decreases to a value below one. These properties result in a better scalability of the method with the dimension of probability space compared to tensor product methods. The refinement is stopped when a global error convergence criterion is reached. The statistical moments are integrated from the response surface by using Monte Carlo sampling of the piecewise polynomial approximation, which is fast to evaluate. For the infinite parameter range of the normal distribution the enclosing hypercube of this Monte Carlo sample is used for the initial sampling.

4.2. Stochastic Collocation

The tensor product Stochastic Collocation (SC) scheme is applied to compute the mean, variance, and the probability density function of $C_1$, $C_d$, and $C_m$ for the uniform input distribution for comparison with SESC results. The main idea behind Stochastic Collocation is to sample the quantity of interest at particular points in the parameter space. Integral statistics such as mean and standard deviation are then computed using quadrature or cubature rules. Non-integral statistics in the form of probability density functions are obtained from the Monte Carlo samples of the Lagrange interpolation of the quantity of interest over the sample points. In the case of a one-dimensional parameter space, these points are quadrature abscissas selected according to the probability measure of the random variable. Tensor product and sparse grid constructions are then utilized to generate the multi-dimensional abscissas.

The choice of one-dimensional abscissas defines the properties of the interpolation formula or the integration rule. Common choices of these nodes are the Clenshaw–Curtis (CC) and Gaussian abscissas. The Clenshaw–Curtis abscissas are the extrema of the Chebyshev polynomials in the interval $[-1, 1]$, see Figure 9 for a two-dimensional example. For any choice of $m > 1$, these points are given by
which renders to a nested rule in the sense that the set of lower order quadratures abscissas for \( m = 2^i + 1 \) is a subset of that of a higher order one with \( m = 2^j + 1 \) for integer values \( i < j \). We adopt the CC rule in this work, since this hierarchical sampling property allows for reusing the samples when increasing the order. The response statistics are then computed as follows. Let \( f(y_1, \ldots, y_d) \) be the quantity of interest as a function of independent random variables \( y_1, \ldots, y_d \) each with a probability density function \( \rho_k : \Omega \rightarrow \Gamma_k \subset \mathbb{R}, k = 1, \ldots, d \), and let

\[
\Lambda := \{y_1^1, \ldots, y_1^{m_1}\} \times \cdots \times \{y_d^1, \ldots, y_d^{m_d}\},
\]

denote the set of multi-dimensional abscissas constructed by taking the tensor product of one-dimensional abscissas corresponding to each random variable \( y_k \). The integral-valued statistics of \( f \) can be approximated using the multi-dimensional quadrature integration formula

\[
\mathbb{E}[g(f)] = \int_{\Gamma_1} \cdots \int_{\Gamma_d} g(f)(y_1, \ldots, y_d) \prod_{k=1}^{d} \rho_k(y_k) dy_1 \cdots dy_d \approx \sum_{j_1=1}^{m_1} \cdots \sum_{j_d=1}^{m_d} g(f)(y_1^{j_1}, \ldots, y_d^{j_d}) w_1^{j_1} \cdots w_d^{j_d},
\]

where \( w_k^{j_k} \) is the weight associated with the quadrature point \( y_k^{j_k} \) and \( g \) is a function defining the desired statistics of \( f \). Clearly, the accuracy of the approximation (6) depends on the location \( y_k^{j_k} \) of each abscissa as well as the total number of points \( m_k \) along each direction \( k \). Notice that cardinality of \( \Lambda \) increases exponentially fast with respect to the number of random variables \( d \) and \( m_k \). The tensor product collocation scheme is therefore not computationally efficient for systems with large number of random inputs.

Alternatively, one can first construct an interpolant of \( f \) from samples in \( \Lambda \) and then estimate the integral and non-integral statistics of \( f \) using Monte Carlo samples of the interpolant. For this purpose,
typically, a multi--dimensional Lagrange interpolation is adopted. More specifically, let
\[ l^k_i(y_k) = \prod_{i_k=1, i_k \neq j}^{m_k} \frac{y_k - y^j_k}{y^j_k - y^k_j}, \]  
(7)
represent the one--dimensional Lagrange polynomials corresponding to abscissa \( y^j_k \), then the interpolant \( I(f) \) of \( f \) is given by
\[ I(f)(y_1, \ldots, y_d) = \sum_{j_1=1}^{m_1} \cdots \sum_{j_d=1}^{m_d} f(y^{j_1}_1, \ldots, y^{j_d}_d) (l^1_{j_1} \otimes \cdots \otimes l^d_{j_d})(y_1, \ldots, y_d), \]  
(8)
where \( \otimes \) denotes the Kronecker product. The desired statistics are approximated from samples of \( I(f) \) by
\[ \mathbb{E}[g(f)] \approx \frac{1}{N} \sum_{i=1}^{N} g(f)(y^{i_1}_1, \ldots, y^{i_d}_d), \]  
(9)
where \( y^{i_1}_1, \ldots, y^{i_d}_d, i = 1, \ldots, N, \) are independent and identically distributed samples of \( y_1, \ldots, y_d \). More information about the technical details of the Stochastic Collocation method can be found in [2, 10, 12, 22].

### 4.3. Padé–Legendre approximation

The Padé–Legendre (PL) method [3] is introduced in this section in a two--dimensional setting, which can be generalized to higher dimensions. The isotropic case with the same number of data points in each direction is considered on a tensor grid of \((N+1) \times (N+1)\) points. In the PL method the approximation of the solution \( u \) is represented as a rational function of the uncertain variables. Overshoots of global approximations at discontinuities can then be reduced by transforming the response into a smooth function through optimizing the denominator. The Padé response surface \( R(u) \) is constructed here on the combination of physical and stochastic spaces
\[ R(u)(x, y) = \frac{P(x, y)}{Q(x, y)} = \frac{\sum_{i=0}^{M} \hat{p}_i \Phi_i(x, y)}{\sum_{i=0}^{L} \hat{q}_i \Phi_i(x, y)}, \]  
(10)
where \( M \) and \( L \) are the orders of the numerator and denominator, respectively. The basis functions \( \Phi_i \) are Legendre polynomials, and \( x \) and \( y \) are either physical or stochastic variables. An accurate approximation to \( u \) is obtained by minimizing the linear Padé approximation error
\[ v_i = \langle P - Qu, \Phi_i \rangle_N, \]  
(11)
for all two--dimensional polynomial bases \( \Phi_i \) of total degree at most \( N \). In multiple dimensions it is generally impossible to require that \( v_i \) vanishes for all \( \Phi_i \). Therefore, \( v_i = 0 \) is imposed only for all polynomial bases of total degree at most \( M \) and \( v_i \) is minimized in the least--squares sense for polynomial bases of total degree from \( M+1 \) to \( M+K \) for some positive integer \( K \). In order to calculate the coefficients in (10), first \( \hat{q}_i \) is calculated from the system of equations
\[ \begin{bmatrix} \langle u\Phi_1, \Phi_{c(M)+1} \rangle_N & \cdots & \langle u\Phi_{c(L), \Phi_{c(M)+1} \rangle_N \\ \vdots & \ddots & \vdots \\ \langle u\Phi_1, \Phi_{c(M+K)} \rangle_N & \cdots & \langle u\Phi_{c(L), \Phi_{c(M+K)} \rangle_N \\ \end{bmatrix} \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_c(L) \end{bmatrix} = 0, \]  
(12)
where \(c(s) = (s+1)(s+2)/2\). The matrix–vector product on the right hand side of (12) is a column vector of \(v_i\) for the polynomial bases of total degree from \(M+1\) to \(M+K\). This system of equations is over–constrained given that \(c(M+K) - c(M) > c(L)\). The optimal solution for \(\hat{q}_i\) in the least–squares sense can be obtained by using the singular value decomposition [18] of the matrix on the left hand side of (12). The computation of the numerator coefficients \(\hat{p}_i\) is then similar to calculating collocation coefficients for \(Qu\)

\[
\hat{p}_i = \frac{\langle P, \Phi_i \rangle_N}{\langle \Phi_i, \Phi_i \rangle_N} = \frac{\langle Qu, \Phi_i \rangle_N}{\langle \Phi_i, P_i \rangle_N} \quad i = 1, 2, ..., c(M). \tag{13}
\]

The PL approximant for \(u\) is finally obtained by substituting \(\hat{q}_i\) from (12) and \(\hat{p}_i\) (13) into (10). For a more theoretical discussion of the PL method is given in [3].

4.4. Inverse Distance Weighting mesh deformation for geometric uncertainties

The treatment of geometric uncertainties poses particular challenges to uncertainty quantification methods. Approaches for dealing with geometric variations of the domain have been proposed based on for example domain transformations [23] and Lagrangian multiplier formulations in fictitious domain discretizations [14]. A more practical treatment of random geometries is obtained by using an automatic mesh deformation method to adapt the computational grid to the variations in the geometry. Here an explicit mesh deformation method [21] is employed which is based on Inverse Distance Weighting (IDW) interpolation of the boundary displacements to the interior of the flow domain. This general purpose mesh motion algorithm does not require solving a system of equations to compute the deformed state of the spatial grid.

Inverse Distance Weighting interpolation [16] is an explicit method for multivariate interpolation of scattered data points. The interpolation surface \(w(\mathbf{x})\) through \(n\) centers \(\mathbf{v} = \{v_1, \ldots, v_n\}\) of the known boundary displacements \(u(\mathbf{x})\) with \(v_i \equiv u(x_i)\) is given in Inverse Distance Weighting by

\[
w(\mathbf{x}) = \frac{\sum_{i=1}^{n} v_i \phi(r_i)}{\sum_{i=1}^{n} \phi(r_i)}, \tag{14}\]

with weighting function

\[
\phi(r) = r^{-c}, \tag{15}\]

where \(r_i = \|\mathbf{x} - x_i\| \geq 0\) is the Euclidian distance between \(\mathbf{x}\) and data point \(x_i\), and \(c = 4\) is a power parameter. The displacements of the internal flow points \(\mathbf{x}\) is then given by the interpolation \(w(\mathbf{x})\) of the boundary node displacements \(\mathbf{v}\) with respect the nominal geometry. The nose region of the resulting meshes for the minimum and maximum thickness for the uniform distribution, \(t/c = 0.1124\) and \(t/c = 0.1298\), respectively, show a good grid quality in Figure 10.

5. Uncertainty quantification results

The mean and standard deviation of the pressure part of the lift \(C_l\), drag \(C_d\), and pitching moment \(C_m\) coefficients computed using Simplex Elements Stochastic Collocation are reported in section 5.1. The relative importance of the three random input parameters is assessed in section 5.2. In section 5.3 the corresponding probability density functions are presented. Results for the three–dimensional uniform distribution are compared with those of the Stochastic Collocation method. In addition the mean and uncertainty bars of the surface pressure coefficient \(C_p\) are also compared with those of the Padé–Legendre
method in section 5.4. Finally in section 5.5 the mean and standard deviation fields of the static pressure are given. In most sections the effect of a uniformly distributed Mach number $M_\infty$ is considered separately before analyzing the combination of the three random parameters with uniform and normal distributions. The UQ methods are compared based on the accuracy of the probabilistic results and the computational costs in terms of the number of deterministic flow solutions.

5.1. Statistical moments

The convergence of SESC for the statistical moments of $C_l$, $C_d$, and $C_m$ is considered in this section for the uniform and normal distribution separately. SC results for the uniform distribution are presented as a comparison.

5.1.1. Uniform distribution

The discussion of the results focuses on the three–dimensional probability space for the combination of the random Mach number $M_\infty$, angle of attack $\alpha$, and thickness–to–chord ratio $t/c$. First the isolated effect the variation in Mach number is however considered independently in order to inspect how the smoothness of the resulting one–dimensional response surface determines the convergence of SESC.

Random Mach number  The response surfaces in terms of the outputs of interest $C_l$, $C_d$, and $C_m$ as function of the random parameter $M_\infty$ are given in Figure 11. The SESC approximation is shown for 10 deterministic samples denoted by the dots. Although the physical problem contains a discontinuity in the flow field in the from of a transonic shock wave, the response surfaces for the integral quantities of interest are relatively smooth. Low order approximations can therefore be expected to result in sufficient accuracy.
(a) Lift coefficient $C_l$

(b) Drag coefficient $C_d$

(c) Pitching moment coefficient $C_m$

Figure 11: Response surfaces of SESC for uniformly distributed Mach number $M_\infty$. 
This is confirmed by the SESC results for the mean $\mu$ and standard deviation $\sigma$ of the force and moment coefficients in Figure 12. The figures show the convergence of the moment approximations as function of an increasing number of samples. The three lines represent the results of limiting the polynomial interpolation order to first $p = 1$, second $p = 2$, and third $p = 3$ degree. A number of samples of $n_s = 5$ deterministic computations is in most cases sufficient to obtain a converged approximation of the statistics. The piecewise linear interpolation for $p = 1$ requires more samples as it converges slower.

The numerical values for the mean and standard deviation of $C_l$, $C_d$, and $C_m$ are given in Table 2 for $p = 3$ and $n_s = \{3, 4, 5, 10\}$. The table shows the fast convergence for the mean value, which is for $n_s = 3$ samples already highly accurate. The difference of the mean values with respect to the deterministic value in the right column of Table 1 is less than 0.5%. The standard deviation shows slightly more variation in the convergence to values that are more than an order of magnitude smaller than the results for the mean.

### Three random parameters

The nonlinear effect of the combination of the three random parameters $M_\infty$, $\alpha$, and $t/c$ is computed here for the uniform distribution using SESC and SC. The convergence plot for the statistical moments in Figure 13 shows the results for SESC up to $n_s = 100$ samples and for SC up to $n_s = 125$. Both methods give a fast convergence to comparable values for the statistics of the smooth three–dimensional response surfaces. The results of SESC show its flexibility in the ability to use an arbitrary number of samples while reusing all previous samples. This leads to detailed convergence information in which the noise is caused by the randomized element refinement and small variations in the iteration residuals of the deterministic computations. The Stochastic Collocation formulation based on Clenshaw–Curtis quadrature points also reuses all previous samples. However, the fast increase of the number of samples in the nested levels in three dimensions from $1^3$ to $3^3 = 27$ and $5^3 = 125$ results in limited choice of the sampling size. The sparse grid Stochastic Collocation formulation would reduce the number of sampling points.

The values for the statistical moments predicted by SESC and SC are given in Tables 3 and 4, respectively. The mean values for SESC in Table 3 show a slower convergence than in the one–dimensional case of Table 2. The standard deviation for $C_l$ and $C_d$ is significantly larger for the three random parameters than for random $M_\infty$. For $C_m$ the standard deviation has a similar value, which suggests that $M_\infty$ is the dominating parameter for variations in the moment coefficient.

Table 4 with the results of SC also includes the statistics based on a Clenshaw–Curtis sampling mesh.

<table>
<thead>
<tr>
<th>$n_s$</th>
<th>mean $C_l$</th>
<th>std. $C_l$</th>
<th>mean $C_d$</th>
<th>std. $C_d$</th>
<th>mean $C_m$</th>
<th>std. $C_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>7.850·10^{-7}</td>
<td>4.971·10^{-3}</td>
<td>6.219·10^{-2}</td>
<td>2.506·10^{-3}</td>
<td>-9.121·10^{-2}</td>
<td>2.471·10^{-3}</td>
</tr>
<tr>
<td>4</td>
<td>7.855·10^{-1}</td>
<td>5.330·10^{-3}</td>
<td>6.219·10^{-2}</td>
<td>2.524·10^{-3}</td>
<td>-9.121·10^{-2}</td>
<td>2.518·10^{-3}</td>
</tr>
<tr>
<td>5</td>
<td>7.855·10^{-1}</td>
<td>5.325·10^{-3}</td>
<td>6.219·10^{-2}</td>
<td>2.523·10^{-3}</td>
<td>-9.121·10^{-2}</td>
<td>2.521·10^{-3}</td>
</tr>
<tr>
<td>10</td>
<td>7.856·10^{-1}</td>
<td>5.407·10^{-3}</td>
<td>6.219·10^{-2}</td>
<td>2.523·10^{-3}</td>
<td>-9.120·10^{-2}</td>
<td>2.522·10^{-3}</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n_s$</th>
<th>mean $C_l$</th>
<th>std. $C_l$</th>
<th>mean $C_d$</th>
<th>std. $C_d$</th>
<th>mean $C_m$</th>
<th>std. $C_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>7.768·10^{-1}</td>
<td>2.426·10^{-2}</td>
<td>6.122·10^{-2}</td>
<td>4.894·10^{-3}</td>
<td>-9.010·10^{-2}</td>
<td>2.463·10^{-3}</td>
</tr>
<tr>
<td>20</td>
<td>7.823·10^{-1}</td>
<td>2.262·10^{-2}</td>
<td>6.191·10^{-2}</td>
<td>4.921·10^{-3}</td>
<td>-9.061·10^{-2}</td>
<td>2.500·10^{-3}</td>
</tr>
<tr>
<td>30</td>
<td>7.819·10^{-1}</td>
<td>2.336·10^{-2}</td>
<td>6.191·10^{-2}</td>
<td>4.921·10^{-3}</td>
<td>-9.066·10^{-2}</td>
<td>2.550·10^{-3}</td>
</tr>
<tr>
<td>100</td>
<td>7.826·10^{-1}</td>
<td>2.309·10^{-2}</td>
<td>6.192·10^{-2}</td>
<td>4.919·10^{-3}</td>
<td>-9.067·10^{-2}</td>
<td>2.533·10^{-3}</td>
</tr>
</tbody>
</table>

Table 2: Statistical moments of SESC for uniformly distributed Mach number $M_\infty$ and $p = 3$.  

Table 3: Statistical moments of SESC for the uniform distribution and $p = 3$.  

The values for the statistical moments predicted by SESC and SC are given in Tables 3 and 4, respectively. The mean values for SESC in Table 3 show a slower convergence than in the one–dimensional case of Table 2. The standard deviation for $C_l$ and $C_d$ is significantly larger for the three random parameters than for random $M_\infty$. For $C_m$ the standard deviation has a similar value, which suggests that $M_\infty$ is the dominating parameter for variations in the moment coefficient.
(a) Mean lift coefficient $C_l$

(b) Standard deviation lift coefficient $C_l$

(c) Mean drag coefficient $C_d$

(d) Standard deviation drag coefficient $C_d$

(e) Mean moment coefficient $C_m$

(f) Standard deviation moment coefficient $C_m$

Figure 12: Statistical moments of SESC for uniformly distributed Mach number $M_{\infty}$. 
Figure 13: Statistical moments of SESC for the uniform distribution.
of 9³ samples which corresponds to 729 deterministic computations. By comparing the results of SESC for \( n_s = 30 \) with those of SC for \( n_s = 27 \) it can be seen that SESC and SC show similar convergence behavior in this case. The largest difference between the SESC predictions and the SC benchmark result with \( n_s = 729 \) is 1.75% for the standard deviation of the pitching moment coefficient.

### 5.1.2. Normal distribution

The normal probability distribution input is intended to test the ability of the applied uncertainty quantification methods to handle unbounded parameter ranges and non–uniform weighting. The results of SESC for \( C_l, C_d, \) and \( C_m \) subject to the three normally distributed random parameters \( M_{\infty}, \alpha, \) and \( t/c \) are summarized in Table 5. In general the statistical moments are comparable with those for the uniform distribution of Table 3, since the mean and standard deviation of the input are identical. The mean and standard deviation match those for the uniform distribution within one percent with the exception for the standard deviation of the pitching moment \( C_m \) which differs 3.79%. Stochastic Collocation could also be applied to the normal distribution, however, in that case Gauss–quadrature sampling is more appropriate than the nested Clenshaw–Curtis rule.

### 5.2. Relative importance of uncertain parameters

For design purposes, it is often desired to rank the importance of the coordinates of a function of interest. This information can then be used to select the most significant input parameters for a more detailed analysis or for investing resources in reducing these input uncertainties. The relative importance is here determined for SESC based on the one–dimensional effects of the parameters. For SC a global sensitivity approach is followed to establish a hierarchy of the input parameters.

#### 5.2.1. Simplex Elements Stochastic Collocation

The standard deviation of the one–dimensional effect of the three parameters separately is used as measure of relative importance for SESC. Although this in a non–local approach, it neglects the nonlinear combined effect of the random parameters in determining their impact. The one–dimensional response
Table 6: Standard deviation importance measure of the uncertain parameters for $C_1$ of SESC for the uniform distribution and $p = 1$.

<table>
<thead>
<tr>
<th>$n_s$</th>
<th>$M_\infty$</th>
<th>$\alpha$</th>
<th>$t/c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.218 · 10^{-2}</td>
<td>1.884 · 10^{-2}</td>
<td>1.410 · 10^{-2}</td>
</tr>
<tr>
<td>20</td>
<td>0.666 · 10^{-2}</td>
<td>1.194 · 10^{-2}</td>
<td>1.803 · 10^{-2}</td>
</tr>
<tr>
<td>30</td>
<td>0.748 · 10^{-2}</td>
<td>1.654 · 10^{-2}</td>
<td>1.630 · 10^{-2}</td>
</tr>
<tr>
<td>100</td>
<td>0.626 · 10^{-2}</td>
<td>1.489 · 10^{-2}</td>
<td>1.756 · 10^{-2}</td>
</tr>
</tbody>
</table>

Table 7: Standard deviation importance measure of the uncertain parameters for $C_3$ of SESC for the uniform distribution and $p = 1$.

<table>
<thead>
<tr>
<th>$n_s$</th>
<th>$M_\infty$</th>
<th>$\alpha$</th>
<th>$t/c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>3.359 · 10^{-3}</td>
<td>3.535 · 10^{-3}</td>
<td>2.518 · 10^{-3}</td>
</tr>
<tr>
<td>20</td>
<td>2.593 · 10^{-3}</td>
<td>2.805 · 10^{-3}</td>
<td>3.089 · 10^{-3}</td>
</tr>
<tr>
<td>30</td>
<td>2.819 · 10^{-3}</td>
<td>3.336 · 10^{-3}</td>
<td>2.876 · 10^{-3}</td>
</tr>
<tr>
<td>100</td>
<td>2.519 · 10^{-3}</td>
<td>3.168 · 10^{-3}</td>
<td>3.033 · 10^{-3}</td>
</tr>
</tbody>
</table>

surfaces are constructed without performing new samples, but by resampling the three dimensional response surface in the direction of a stochastic coordinate axis while keeping the other two parameters fixed at their mean value.

The resulting standard deviations for the uniform input distribution are given in Tables 6 to 8 for the parameters $M_\infty$, $\alpha$, and $t/c$, and the coefficients $C_1$, $C_3$, and $C_m$ with $n_s = \{10, 20, 30, 100\}$ and $p = 1$. The equivalent results for the normal distribution are qualitatively identical. Conclusions are drawn from this data in a comparison with the SC results in the next section.

5.2.2. Stochastic Collocation

In a deterministic setting a ranking of inputs is typically obtained from a sensitivity analysis where the derivatives of the function with respect to its coordinates are computed at the parameter values. A direct extension of this approach for the case of functions of random variables is the so-called Morris method [11]. In this approach, derivatives of the function referred to as the elementary effects (EE) are computed at random points in the parameter space

$$EE_k := \frac{f(y_1, \ldots, y_{k-1}, y_k + \Delta, y_{k+1}, \ldots, y_d) - f(y_1, \ldots, y_{k-1}, y_k, y_{k+1}, \ldots, y_d)}{\Delta}$$

(16)

where $y_1, \ldots, y_d$ are random samples of $y_1, \ldots, y_d$, respectively, and $\Delta$ is a small number. In contrast to sensitivities linearized around the deterministic conditions, the mean value $\mu_{EE}$ of EE gives a weighted sensitivity over the entire probability space, which takes into account the non–linear combined effect

Table 8: Standard deviation importance measure of the uncertain parameters for $C_m$ of SESC for the uniform distribution and $p = 1$.

<table>
<thead>
<tr>
<th>$n_s$</th>
<th>$M_\infty$</th>
<th>$\alpha$</th>
<th>$t/c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
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<td>0.536 · 10^{-3}</td>
<td>0.142 · 10^{-3}</td>
</tr>
<tr>
<td>20</td>
<td>2.303 · 10^{-3}</td>
<td>0.955 · 10^{-3}</td>
<td>0.393 · 10^{-3}</td>
</tr>
<tr>
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<td>2.448 · 10^{-3}</td>
<td>0.584 · 10^{-3}</td>
<td>0.405 · 10^{-3}</td>
</tr>
<tr>
<td>100</td>
<td>2.390 · 10^{-3}</td>
<td>0.568 · 10^{-3}</td>
<td>0.450 · 10^{-3}</td>
</tr>
</tbody>
</table>
Table 9: Global sensitivity importance measure of the uncertain parameters of SC for the uniform distribution.

<table>
<thead>
<tr>
<th></th>
<th>$C_l$</th>
<th>$C_d$</th>
<th>$C_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_\infty$</td>
<td>$0.494 \cdot 10^{-2}$</td>
<td>$2.442 \cdot 10^{-3}$</td>
<td>$2.387 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$1.338 \cdot 10^{-2}$</td>
<td>$2.949 \cdot 10^{-3}$</td>
<td>$0.263 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>$t/c$</td>
<td>$1.710 \cdot 10^{-2}$</td>
<td>$2.995 \cdot 10^{-3}$</td>
<td>$0.022 \cdot 10^{-3}$</td>
</tr>
</tbody>
</table>

Table 10: Ranking of relative importance of the uncertain parameters.

<table>
<thead>
<tr>
<th></th>
<th>$C_l$</th>
<th>$C_d$</th>
<th>$C_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_\infty$</td>
<td>3</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>2</td>
<td>1–2</td>
<td>2</td>
</tr>
<tr>
<td>$t/c$</td>
<td>1</td>
<td>1–2</td>
<td>3</td>
</tr>
</tbody>
</table>

of the parameters. An approximate measure of importance of the input distributions is obtained by multiplying the absolute value of $\mu_{EE}$ with the standard deviation of the input

$$|\mu_{EE}|\sigma,$$

for $M_\infty$, $\alpha$, and $t/c$. The results for $n_s = 729$ given in Table 9 show good agreement with the SESC measure. Since the two importance measures are extracted from two different data sets for SESC and SC using two different approaches, their match gives additional confidence in the validity of the derived conclusions.

The SESC results in Tables 6 to 8 and the SC data in Table 9 give an identical ranking of relative importance of the random input parameters for the outputs of interest $C_l$, $C_d$, and $C_m$ given in Table 10. For the lift and the pitching moment, $t/c$ and $M_\infty$ are the dominating parameters, respectively, and $\alpha$ is second most important in both cases. The angle of attack and the airfoil thickness both have an importance of the same order of magnitude for $C_d$, while $M_\infty$ has a smaller effect. The general conclusion is that the effect of none of the random parameters is negligible compared to the other parameters based on the considered outputs of interest.

5.3. Probability density functions

The probability densities of $C_l$, $C_d$, and $C_m$ give more detailed insight into the effect of the random parameters on the force and moment coefficients. The probability density functions (PDF) are computed using SESC for the uniform and normal input distribution. Results for the uniform distribution are compared with those of Stochastic Collocation.

5.3.1. Uniform distribution

The PDFs of the lift, drag, and moment coefficients of SESC and SC are given in Figure 14 for the uniform distribution. For SESC the results are shown for $n_s = \{10, 20, 30, 100\}$ samples and SC is considered for $n_s = \{27, 125, 729\}$ samples. The output PDFs have a clear non–uniform character which is to be expected from a multi–dimensional uniform input. The results of SESC and SC have a reasonably good agreement. The PDFs of SESC show more variation between the different curves than the SC results, which are practically converged for $n_s = 27$. SESC gives a significant improvement from $n_s = 10$ to $n_s = 20$ samples. Further increasing the number of samples to $n_s = 30$ and $n_s = 100$ results only in small differences, except for the moment coefficient $C_m$. 
Figure 14: Probability densities for the uniform distribution.
5.3.2. Normal distribution
The results for the propagation of the normal distribution using SESC in Figure 15 show remarkably similar PDFs for $C_l$ and $C_d$ to those for the uniform input distribution. This explains the small differences in the statistics for the uniform and normal input distributions. The PDFs for $n_s = \{10, 20, 30, 100\}$ give a larger spread than for the uniform case, which can be reduced using a more advanced element refinement criterion than the probability measure.

5.4. Surface pressure coefficient uncertainty bars
One of the objectives of UQ is to represent the effect of physical variability in terms of uncertainty bars in the presentation of numerical results similar to the documentation standards for experimental data. In this section the mean and uncertainty bars of the surface pressure coefficient $C_p$ are presented. SESC is again used for the uniform and normal distribution, of which the results for the uniform distribution are compared with those of SC. The Padé–Legendre (PL) method is also employed here for a uniformly distributed Mach number $M_\infty$ to capture the high gradients in the response surface of $C_p$ at the shock location in physical space.

5.4.1. Uniformly distributed Mach number
The PL method is considered here since the shock wave in physical space is expected to result in large gradients in the $C_p$ response surface. For the SC method this could lead to an oscillatory solution. The PL approach has been shown to reduce the overshoots due to an oscillatory response approximation using a rational approximation of the output quantity.

A combination of the random dimension for $M_\infty$ and the curvilinear coordinate along the airfoil surface in physical space is considered for the PL expansion. The two–dimensional Stochastic Collocation response surface is then constructed using $n_s = 9$ deterministic samples and partitioning of the airfoil surface in a number of spatial elements. The PL post–processing successfully reduces the oscillations as shown in Figure 16 for the 99% uncertainty bars along the airfoil surface. The length of the uncertainty bars is dominated by the surface pressure variations caused by the changing shock wave location. The uncertainty band represents this change in location of the shock wave and indicates small variations in shock wave strength. A small undershoot is visible in the shock region. The uncertainty bars show also local maxima at the leading edge and on the lower surface at the location of maximum airfoil thickness. In the mean sense the discontinuity is smeared out with respect to the deterministic pressure distribution of Figure 7a due the varying shock wave location.

5.4.2. Three random parameters
The uncertainty bars on the mean $C_p$ of SESC as a result of the three random parameters with a uniform and normal distribution are given in Figure 17. The uncertainty bars for 90%, 95%, and 99% are shown to demonstrate that the method maintains robustness for representing increasing percentiles. The convergence is assessed by plotting the results for $n_s = 20$ and $n_s = 30$ samples. The uncertainty bars are significantly larger over large regions of the airfoil surface compared to the uncertainty bars for only random Mach number presented in the previous section, which indicates a more global effect of the combined randomness in the three parameters. The results for $n_s = 20$ and $n_s = 30$ show good agreement.
Figure 15: Probability densities of SESC for the normal distribution.
with obviously larger uncertainty bars for higher percentiles and without overshoots at the shock wave as a consequence of the extremum diminishing property.

Results for the normal distribution show a region of larger uncertainty bars upstream of the shock location due to the overprediction of the effect of the samples in the vertices of the enclosing hypercube by the probability refinement of the elements. This effect decreases with increasing sample size. The length of the uncertainty bars increase faster with increasing percentiles compared due to the uniform random parameters due to the tails of the normal distribution.

Results for the moderate 95% uncertainty bars and the uniform input distribution are compared with those of SC in Figure 18 with $n_s = \{27, 125, 729\}$ samples. The SC data shows a good agreement with the SESC results. Small oscillations are present in the approximation with $n_s = 27$, of which the overshoots decrease with increasing polynomial order, since the weak transonic shock wave at the surface does not lead to a strong discontinuity in probability space.

5.5. Mean and standard deviation pressure field

Although the integral quantities in problems involving discontinuities can be smooth functions of the random parameters, it is often necessary to compute also the mean and standard deviation fields of local flow quantities to understand the mechanism of amplification of input uncertainty. This requires application of the UQ post-processing to the nodal flow conditions in all volumes in the spatial discretization, of which the response surfaces contain a discontinuity in the shock region. SESC is used in this section to generate the mean and standard deviation fields of the static pressure for a uniformly distributed Mach number $M_\infty$ and the combination of three normal distributions.
Figure 17: Mean surface pressure coefficient $C_p$ and uncertainty bars of SESC.
5.5.1. **Uniformly distributed Mach number**

The mean pressure field presented in Figure 19 based on $n_s = 10$ samples for the uniformly distributed $M_{\infty}$ clearly visualizes the smearing of the shock wave in the mean sense in comparison with the deterministic pressure solution of Figure 7(b). The standard deviation field shows the localized production of standard deviation in the shock region due to the variation in the location of the shock wave. The maximum standard deviation is found around the center of shock location variation within the flow field away from the wall due to the absence of the shock wave in the subsonic boundary layer. This contrasts with earlier findings of a maximum standard deviation at the surface for inviscid Euler computations [19].

The fields correspond on the airfoil surface to the case of the mean and uncertainty bars of $C_p$ of the PL method in Figure 16. The result confirms the conclusions from the application of the PL method that the uncertainty is dominated by the standard deviation in the shock region. The extended region of higher standard deviation at the wall downstream of the shock wave can be recognized in both results. Also the minor features present in the uncertainty bar distribution in Figure 16 are matched by the SESC predictions although not visible due to the scale of the color map. The noise in the contour lines is caused by the post-processing of the mesh partitioning over several processors and other slight mesh imperfections.

5.5.2. **Normal distribution**

Finally the mean and standard deviation field for the three random parameters with a normal distribution are shown in Figure 20 for $n_s = 20$ samples. The statistics are again computed by applying SESC to the sampled results in each spatial volume. Since the geometric uncertainty in the thickness–to–chord ratio $t/c$ is treated by mesh deformation, the volume locations do however not coincide exactly in physical space for the different samples. The results can be interpret as an approximation of the statistics fields scaled back to the mesh for the mean airfoil thickness. This approach is appropriate here due to the small variations in the airfoil thickness, although it can slightly affect the contour lines.
The standard deviation at the shock is not as dominant in this case as for the uniform Mach number results discussed in the previous section. The variation shows two more pronounced local maxima at the leading edge and on the lower surface around the point of maximum thickness. These results correspond to similar observations for the surface pressure uncertainty bars of Figures 17(b), 17(d), and 17(f). It can be concluded that the combination of the three parameters has a more non-local effect on the pressure field than uniform Mach number alone. It is therefore important to resolve the combined effect of multiple uncertainties, although the maximum standard deviation for the three normal distributions is smaller compared to uniform $M_{\infty}$.

6. Conclusions

Three non-intrusive Stochastic Collocation methods are compared in application to the transonic RAE 2822 airfoil test case of the NODESIM Workshop on Quantification of CFD Uncertainties with a combination of three uncertainties in the Mach number $M_{\infty}$, angle of attack $\alpha$, and thickness-to-chord ratio $t/c$ described by a uniform and normal distribution. The test problem requires both the efficient discretization of the smooth response surfaces for the integral quantities of the lift $C_l$, drag $C_d$, and pitching moment $C_m$ coefficients in multiple dimensions, and the robust approximation of the discontinuous responses for the surface pressure coefficient $C_p$ and the two-dimensional pressure field $p$. Simplex Elements Stochastic Collocation (SESC) based on higher order interpolation of randomized samples in a simplex elements discretization of probability space is found to be efficient in higher-dimensional probability spaces and robust due to the extremum diminishing (ED) property. Stochastic Collocation (SC) based on nested Clenshaw–Curtis quadrature sampling shows for the uniform distribution comparable performance in this problem with a maximum difference in the standard deviation of $C_m$ of 1.75%, however, with limited choice of the sample size. The Padé–Legendre (PL) formulation of the Stochastic Collocation method is used to construct the uncertainty bars on $C_p$ for a uniformly distributed $M_{\infty}$. The explicit Inverse Distance
Weighting (IDW) mesh deformation method used for handling the geometric uncertainty in \( t/c \) results in good quality of the spatial grids in the deterministic computations.

For the approximation of the mean, standard deviation, and the probability density functions (PDF) of \( C_L, C_D, \) and \( C_M, \) a number of deterministic samples of 20–30 and 27 for SESC and SC, respectively, are in most cases sufficient to obtain reasonable convergence. It was experienced that it is difficult to draw more specific conclusions from the comparison due to the relatively small input uncertainties with respect to a realistic deterministic convergence criterion in combination with the weak transonic shock wave. The relative importance of the parameters determined by one-dimensional standard deviations for SESC and by a global sensitivity measure for SC shows that the effect of none of the parameters is negligible compared to the other parameters for the considered outputs of interest. For the \( C_L \) and \( C_M, \) the dominating parameters are \( t/c \) and \( M_\infty, \) respectively, and \( \alpha \) is second most important in both cases. Both \( \alpha \) and \( t/c \) have an importance of the same order of magnitude for \( C_D, \) while \( M_\infty \) has a smaller effect. SESC gives a robust ED approximation of the uncertainty bars for \( C_p \) for increasing percentiles of 90%, 95%, and 99%, which show a sensitive shock wave location. The uncertainty bars of PL and SC show small overshoots in the shock region which decrease with increasing polynomial order, since the weak transonic shock wave does not lead to a strong discontinuity at the airfoil surface. The mean and standard deviation fields for \( p \) show a maximum production of standard deviation in the shock region away from the wall outside the subsonic boundary layer.

In conclusion the NODESIM workshop has in general been the motivation for the application and comparison of a large number of uncertainty quantification approaches to challenging problems in computational fluid dynamics (CFD) of practical interest. For a possible next workshop it is suggested to start the uncertainty quantification from actually measured physical variations, which could be useful for the necessary validation of uncertainty quantification results. It would also be interesting to direct the uncertainty quantification efforts to answering a specific question through robust design optimization or reliability analysis.
References


